# Chapter 6

# ENERGY CONSIDERATION

**Introduction:**

**1.)** Newton's Second Law is nice because it provides us with a technique for attacking a certain class of problems. "Focus your attention on the *forces* acting on a body," it says, "and you can deduce something about the body's *acceleration*." As useful as this is, there are other ways to approach motion and physical systems. We are about to develop a new perspective that focuses on the *energy content* of a system.

**2.)** One of the techniques theoretical physicists use to characterize a physical system is to identify all the parameters (i.e., force or displacement or whatever) that govern a phenomenon of interest, then multiply those parameters together. The resulting number or vector then acts as a watermark that allows an individual to predict how pronounced the phenomenon in question will be in a particular instance.

**a.)** Example: What governs the change of a body's velocity? The *force component along the line of motion* certainly matters; so does the *distance* over which the force is applied. If the product of those two quantities is big, you know the resulting velocity change will be relatively big. If small, the velocity change will be relatively small.

**3.)** We are about to build a mathematical model that begins with the very product alluded to in *Part 2a*, then look to see where that definition logically takes us. Hold on to your skirts, ladies. This should be fun.

**A.) Work:**

**1.)** The beginning definition: As was said above, a change in a body's velocity is governed by the magnitude of the *component of force along the line of the displacement* and the magnitude of the *displacement* itself. The product of those two parameters, $F_{//}$ and $d$, defines the dot product $\boldsymbol{F}\cdot\boldsymbol{d}$. That quantity is given a special name. It is called *work*.

**2.)** By definition, the *work* $W_F$ done by a constant force $\boldsymbol{F}$ acting on a body that moves some straight-line distance $\boldsymbol{d}$ (note that $\boldsymbol{d}$ is a vector that defines both the direction and the magnitude of the displacement of the body) is equal to:

$$\begin{aligned} W &= \mathbf{F} \cdot \mathbf{d} \\ &= |\mathbf{F}||\mathbf{d}| \cos \phi, \end{aligned}$$

where $\phi$ is the angle between *the line of* $\boldsymbol{F}$ and *the line of* $\boldsymbol{d}$.

**3.)** Example: A box of mass $m = 2\ kg$ moving over a *frictional floor* ($\mu_k$=.3) has a force whose magnitude is $F = 25$ *newtons* applied to it at a $30^o$ angle, as shown in Figure 6.1 (note that $\phi$ equals the angle $\theta$ in the sketch). The crate is observed to move 16 meters in the horizontal before falling off the table (that is, $\boldsymbol{d} = 16\boldsymbol{i}$ *meters*). An f.b.d. for the forces acting on the block is shown in Figure 6.2.

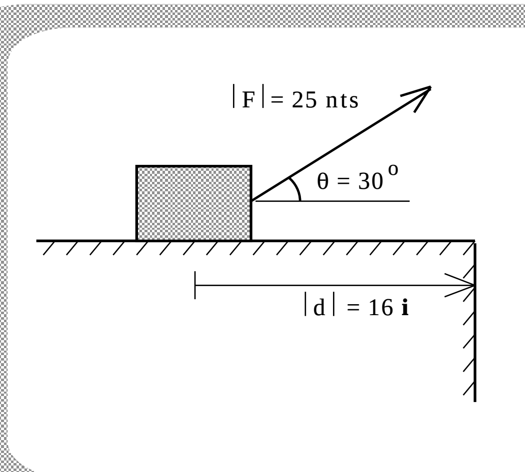


FIGURE 6.1

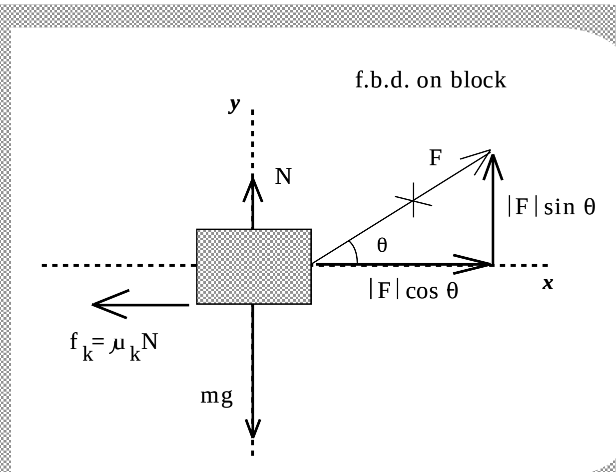


FIGURE 6.2

**a.)** How much work does $\boldsymbol{F}$ do before the crate takes the plunge?

$$\begin{aligned} W_{\mathbf{F}} &= \mathbf{F} \cdot \mathbf{d} \\ &= \quad |\mathbf{F}| \qquad\qquad |\mathbf{d}| \qquad \cos\theta \\ &= (25 \text{ newtons})\ (16 \text{ meters}) \cos 30^o, \\ &= 346.4 \text{ newton-meters.} \end{aligned}$$

**Note 1:** A *newton-meter* (or a $kg{\cdot}m^2/s^2$) is the MKS unit for both work and energy. It has been given a special name--the JOULE. We could, therefore, have written *the work done by* $F$ as "346.4 joules."

**Note 2:** Work and energy units in the CGS system are *dyne-centimeters* (or $gm \cdot cm^2/s^2$). That combination has been given the name ERGES. In the English system, work and energy units are in FOOT-POUNDS.

**b.)** The above *dot product* was done from a polar notation approach (i.e., you multiplied the *magnitude of one vector* times the *magnitude of the second vector* times the *cosine of the angle between the line-of-the-two-vectors*) because the force information was given in polar notation. If the initial information had been given in *unit vector notation*, we would have used the unit vector approach for the dot product.

For the sake of completeness, let us do the problem from that perspective:

**i.)** The *unit vector* representation of the force vector presented in our problem above is:

$$\mathbf{F} = (21.65\,\mathbf{i} + 12.5\,\mathbf{j}) \text{ nts.}$$

**ii.)** *Dot products* executed in *unit vector notation* are defined as:

$$\begin{aligned}\mathbf{F} \cdot \mathbf{d} &= (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}) \cdot (d_x\mathbf{i} + d_y\mathbf{j} + d_z\mathbf{k}) \\ &= (F_x\, d_x) + (F_y\, d_y) + (F_z\, d_z).\end{aligned}$$

**iii.)** As $d_z = 0$, we have:

$$\begin{aligned}W_{\mathbf{F}} &= \mathbf{F} \cdot \mathbf{d} \\ &= (F_x\mathbf{i} + F_y\mathbf{j}) \cdot (d_x\mathbf{i} + d_y\mathbf{j}) \\ &= (21.65\,\mathbf{i} + 12.5\,\mathbf{j}) \cdot (16\,\mathbf{i}) \\ &= (21.65 \text{ newtons})(16 \text{ meters}) + (12.5 \text{ newtons})(0) \\ &= 346.4 \text{ joules.}\end{aligned}$$

This is the same value we determined using the polar approach. As expected, the two approaches yield the same solution.

**c.)** In our example, how much work does the *normal force* do? The temptation is to assume that we need to determine the magnitude of the normal force before doing this, but a little insight will save us a lot of trouble here. From the definition of *work*:

$$\begin{aligned}W_{\mathbf{N}} &= \mathbf{F} \cdot \mathbf{d} \\ &= |\mathbf{N}||\mathbf{d}|\cos\phi.\end{aligned}$$

The trick is to notice that the angle $\phi$ between $\boldsymbol{N}$ and $\boldsymbol{d}$ is $90^o$ (see the *free body diagram* shown in Figure 6.2). As $cos\ 90^o = 0$, $W_N = 0$.

In fact, *normal forces* are always *perpendicular* to a body's motion. As such, their work contribution will *always* be ZERO. <u>*Normal forces* do no work on a moving body.</u>

**d.)** How much work does the *frictional force* do on the body as it moves toward the abyss?

**i.)** To do this part, we need to determine the *normal force* $N$ so that we can determine the *frictional force* using the relationship $f_k = \mu_k N$. Utilizing both the f.b.d. shown in Figure 6.2 and Newton's Second Law:

$$\underline{\sum F_y:}$$
$$N + F(\sin\theta) - mg = ma_y.$$

As $a_y = 0$, rearranging yields:

$$\begin{aligned} N &= -F(\sin\theta) + mg \\ &= -(25\text{ newtons})(\sin 30^o) + (2\text{ kg})(9.8\text{ m/s}^2) \\ &= 7.1\text{ newtons.} \end{aligned}$$

The frictional force will be:

$$\begin{aligned} f_k &= \mu_k N \\ &= (.3)(7.1\text{ newtons}) \\ &= 2.13\text{ newtons.} \end{aligned}$$

**ii.)** Noticing that the angle between the *line of* $\boldsymbol{f_k}$ and the *line of* $\boldsymbol{d}$ is $180^o$, the work done by friction will be:

$$\begin{aligned} W_f &= \mathbf{f}_k \cdot \mathbf{d} \\ &= |\mathbf{f}_k||\mathbf{d}| \cos\phi \\ &= (2.13\text{ newtons})(16\text{ meters}) \cos 180^o \\ &= -34.1\text{ joules.} \end{aligned}$$

**Note 1:** Yes, *work* quantities can be *negative*. In fact, whenever the angle between the *line of* $\boldsymbol{F}$ and the *line of* $\boldsymbol{d}$ is greater than $90^o$ and less than or equal to $180^o$, the cosine of the angle will yield a *negative* number.

**Note 2:** The *negative sign* is not associated with *direction*. Work is a scalar quantity--IT HAS NO DIRECTION. A negative sign in front of a *work quantity* tells you that the force doing the work is oriented so as to *slow the body down*.

**4.)** Comments:

**a.)** Mathematically, the physics concept of *work* is rigidly defined. Hold a 25 pound weight at arm length for fifteen minutes and although there may be sweat pouring off your brow, you will nevertheless be doing no *work*. Why? Because for *work* to occur, a FORCE must be applied to a body *as it moves* over a DISTANCE. If there is no displacement (example: your arm held motionless for fifteen minutes), no *work* is done.

**b.)** When *work* is done by a single force acting on an object, it *changes the object's motion* (i.e., speeds it up or slows it down). Again, the key is motion. Things become more complicated when many forces act on a body, but in all cases, having some net amount of work being done implies there is motion within the system.

**c.)** On an intuitive level, forces that do *positive work* are oriented so as to make a body speed up; forces that do *negative work* are oriented so as to make a body slow down. It is as though doing *positive work* puts energy *into* the system while doing *negative work* pulls energy *out of* the system. (We will more fully define the idea of *energy* shortly).

## B.) *Work* Due to Variable Forces:

**1.)** When *work* is done by a force-and-displacement combination that in some way varies as a body moves (i.e., the force changes magnitude along the way or the angle between the force and displacement changes), we can no longer write $W_F = \mathbf{F}\cdot\mathbf{d}$ and proceed from there. That relationship holds only when all the parameters stay constant throughout the motion.

To deal with the *varying force or angle* situation, we must use Calculus.

**a.)** If we take a differential displacement $d\mathbf{r}$ to be a very small distance traveled along a body's path as it moves from one point to another in a force field $\mathbf{F}$, the *differential amount of work* (read this "a very small amount of work in comparison to the whole of the work done") will equal the dot product between $\mathbf{F}$ and $d\mathbf{r}$. Mathematically, this is:

$$dW = \mathbf{F} \cdot d\mathbf{r}.$$

**b.)** Summing this *dot product* over the entire path will give us the entire amount of work done by the force over the motion. In short, if we want the work done in such cases, we must determine the integral:

$$W = \int dW = \int \mathbf{F} \bullet d\mathbf{r}.$$

**c.)** There are two ways to evaluate a *dot product*: using a *unit vector* approach and using a *polar* approach. The easiest way to see how each works is with an example.

**2.)** Consider a body constrained by a *group of forces* to move along a path described by $y = 1$. If one of those forces (call it $\boldsymbol{F}$) is defined by the function:

$$\mathbf{F} = k(x\mathbf{i} + y\mathbf{j}),$$

where $k$ is a constant equal to *1 nt/meter* (that is, $k$ is in the problem solely for the sake of units), how much work does the force do as the body moves from $x_1 = 1$ *meter* to $x_2 = 5$ *meters?*

**Note 1:** The situation is shown in Figure 6.3, complete with the *force magnitude* and *angle* at the points *(1, 1)* and *(5, 1)*.

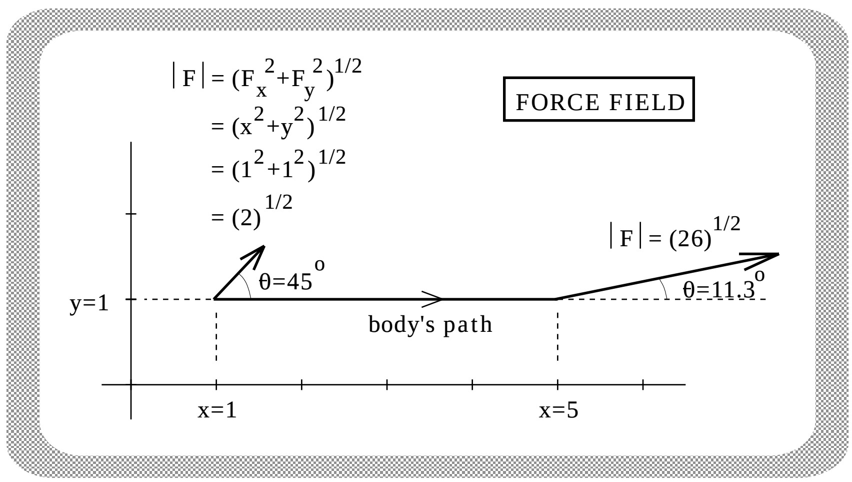


FIGURE 6.3

**Note 2:** The other forces acting on the body also do *work*, but we are not interested in them. All we want is the amount of *work* done by force $\boldsymbol{F}$.

**3.)** Using a UNIT VECTOR approach:

**a.)** The most general way to define a *differential displacement* $d\mathbf{r}$ is with the relationship:

$$d\mathbf{r} = (dx)\mathbf{i} + (dy)\mathbf{j} + (dz)\mathbf{k}.$$

That is, vectorially adding *differential displacements* in the *x, y,* and *z* directions will yield a vector whose net displacement is differential and whose magnitude is:

$$|d\mathbf{r}| = [dx^2 + dy^2 + dz^2]^{1/2}.$$

**b.)** Looking at Figure 6.4, it is clear that the displacement is in the *x* direction, and the *x* direction only. As such, the *magnitude of differential displacement* in this example will simply be *dx*.

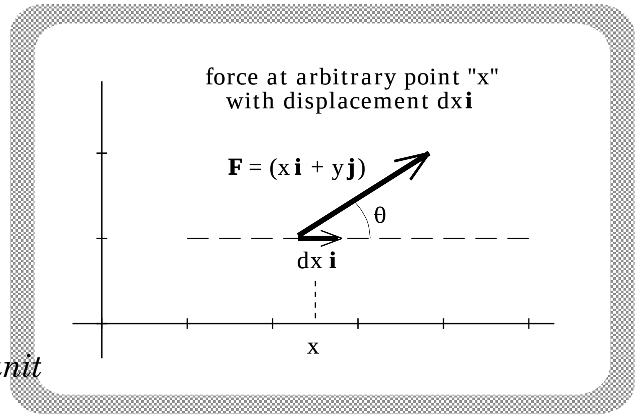


FIGURE 6.4

**c.)** We know the *force function* (i.e., $\boldsymbol{F} = k(x\mathbf{i} + y\mathbf{j})$) in *unit vector notation*--that is the way the force was given in the problem. Remembering how to do a *dot product* in *unit vector notation*, and remembering that $k = 1$, we can write our *work expression* as:

$$\begin{aligned}
W &= \int dW \\
&= \int \mathbf{F} \bullet d\mathbf{r} \\
&= \int (kx\mathbf{i} + ky\mathbf{j}) \bullet (dx\mathbf{i}) \\
&= \int_{x=1}^{5} (kx)dx \\
&= \left[ k\frac{x^2}{2} \right]_{x=1}^{5} \\
&= \left[ (1)\frac{(5)^2}{2} - (1)\frac{(1)^2}{2} \right] \\
&= 12 \text{ joules.}
\end{aligned}$$

**Note:** If you are wondering why the *ky**j*** term has not been carried along in the above calculation, think about the definition of *dot product* in *unit vector notation*. The *x components* are multiplied together, as are the *y components* and *z components*, then the products are added together. As there is no *y-direction differential displacement* (i.e., no *dy*), the *y* contribution to the *dot product* is zero leaving only *kx(dx)* under the integral.

**4.)** Using a POLAR approach:

**a.)** The magnitude of the *differential displacement* is still:

$$|d\mathbf{r}| = dx.$$

**b.)** The magnitude of the *force function* (i.e., $\boldsymbol{F} = k(x\boldsymbol{i} + y\boldsymbol{j})$) is:

$$\begin{aligned} |\mathbf{F}| &= [(kx)^2 + (ky)^2]^{1/2} \\ &= k(x^2 + y^2)^{1/2}. \end{aligned}$$

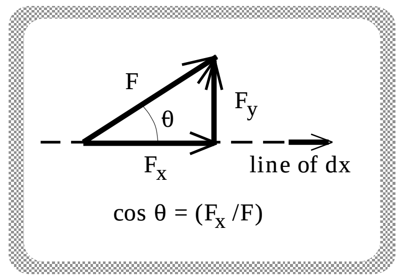

FIGURE 6.5

**c.)** Looking at Figure 6.5, the *cosine* of the angle between *the line of* $\boldsymbol{F}$ and *the line of* $d\boldsymbol{r} = (dx)\boldsymbol{i}$ is:

$$\begin{aligned} \cos\theta &= [F_x]/[F] \\ &= [kx] / [k(x^2 + y^2)^{1/2}] \\ &= [x] / (x^2 + y^2)^{1/2}. \end{aligned}$$

**d.)** Putting it all together:

$$\begin{aligned} W &= \int dW \\ &= \int \mathbf{F} \bullet d\mathbf{r} \\ &= \int \quad |\mathbf{F}| \quad |d\mathbf{r}| \quad \cos\theta \\ &= \int \left[k(x^2 + y^2)^{1/2}\right][dx]\left[\frac{x}{\left[(x^2 + y^2)^{1/2}\right]}\right] \\ &= \int_{x=1}^{5} (kx)dx. \end{aligned}$$

This is the same integral we found using the *unit vector* approach.

## C.) Even *More* Fun With Force Functions That Vary:

**1.)** A ball hangs from a rope attached to a ceiling, as shown in Figure 6.6. A variable, horizontal force $\boldsymbol{F}$ is applied to the ball so that:

**a.)** ***F*** is ALWAYS horizontal; and

**b.)** ***F's*** *magnitude* varies so that the ball moves up the arc with a *constant velocity*; and

**c.)** The ball's *velocity* is very low.

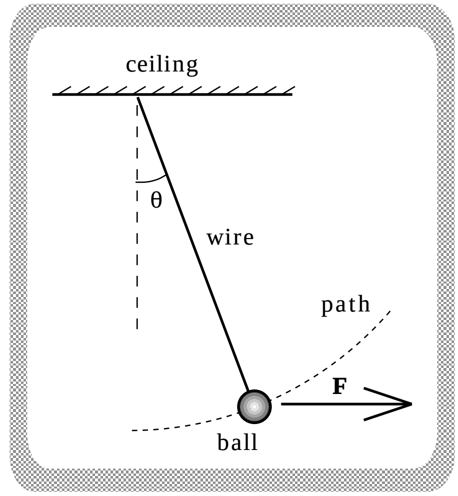


FIGURE 6.6

**2.)** Assuming the ball's mass is $m$, how much *work* does ***F*** do as the ball moves from $\theta = 0^o$ to $\theta = \theta_1$?

**3.)** Using a *POLAR* approach:

**a.)** For a *dot product* using a *polar* approach, we need the *magnitude of the force*, the *magnitude of the displacement*, and the *angle between the line of the force and the line of the displacement*.

**b.)** Because the angle between the force and the displacement is constantly changing, we must use the *integral form* of the definition of *work*. That means the displacement we want is a *differential displacement* $d\mathbf{r}$ and the amount of work done by the force $\boldsymbol{F(\theta)}$ will be:

$$W = \int dW = \int \mathbf{F} \cdot d\mathbf{r}.$$

**Note:** There is a common error people make in doing problems like this. If one determines ***F*** at a known angle (say, $\theta = \theta_1$), the *dot product* under the integral will not be general. As the integral is supposed to sum $\boldsymbol{F \cdot dr}$ terms at all angles between the limits, the *function* that defines ***F*** must be generally good for any arbitrary angle.

**c.)** We need to use N.S.L. to determine the force as a function of the *angular displacement* of the string (i.e., $\boldsymbol{F(\theta)}$) at any arbitrary angle.

**i.)** As we have assumed that the *magnitude of the velocity* is constant, the acceleration of the mass along the path will be zero.

**ii.)** As we have assumed the *magnitude of the velocity* is VERY, VERY SMALL, we can ignore the *centripetal acceleration* (i.e., $v^2/R$) directed along *the line of tension* as the mass moves up the arc.

**iii.)** Note that with these assumptions, there is no appreciable acceleration in ANY direction.

**d.)** We are interested in determining ***F***. As there is no acceleration in any direction, and as ***F*** is in the *horizontal direction*, we can use N.S.L. and the f.b.d. shown in Figure 6.7a to write:

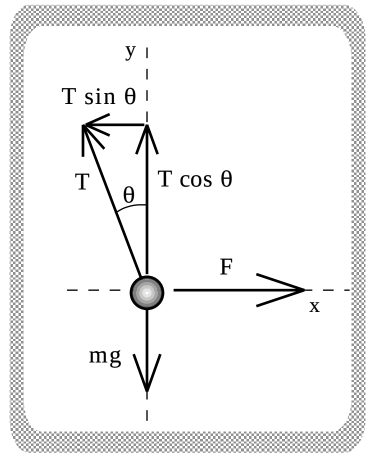


FIGURE 6.7a

$\underline{\sum F_x:}$

$$-T(\sin\theta) + F = ma_x$$
$$= 0 \qquad (\text{as } a_x = 0)$$
$$\Rightarrow \quad F = T(\sin\theta). \qquad (\text{Equation A}).$$

$\underline{\sum F_y:}$

$$T(\cos\theta) - mg = ma_y$$
$$= 0 \qquad (\text{as } a_y = 0)$$
$$\Rightarrow \quad T = mg/(\cos\theta) \qquad (\text{Equation B}).$$

Substituting *Equation B* into *Equation A*, we get:

$$\begin{aligned} F &= T(\sin\theta) \\ &= [mg/(\cos\theta)](\sin\theta) \\ &= mg\,[(\sin\theta)/(\cos\theta)] \\ &= mg(\tan\theta). \end{aligned}$$

**e.)** The *differential displacement d**r*** of the mass requires the string to move in such a way as to subtend a *differential angle* $d\theta$ (see Figure 6.7b). If the angle is truly differential, it will be tiny and the *arc length* of the arc upon which the mass moves will approach the net *differential displacement d**r***. In other words, if we can determine an expression for the arc length associated with $d\theta$, we will have an expression for the magnitude of ***dr***.

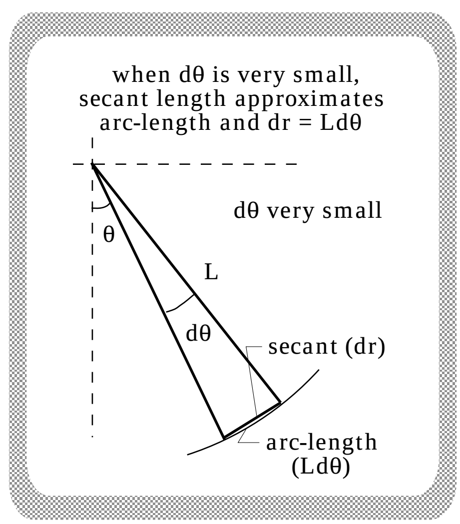


FIGURE 6.7b

If the angle $d\theta$ is measured in *radians*, the arc length will equal *L(*$d\theta$*)*. That means:

$$|d\mathbf{r}| = L(d\theta).$$

**Note:** This is not as obscure as it sounds. A *one-radian angle* is defined as an angle whose *arc length* equals the radius *R* of the circle upon which it is measured. That means a *one-half radian angle* has an *arc length* of *(1/2)R*, and a $\theta$ *radian angle* has an *arc length* of *(*$\theta$*)R*. As the angle in our case is $d\theta$, the *arc length* is the radius (i.e., the string's length *L*) times the angle $d\theta$ or $L\,d\theta$.

**f.)** The only thing needed is to relate the angle between the *line of **F*** and the *line of d**r*** (call this $\phi$ for the moment). AS CAN BE SEEN in Figure 6.7c, $\phi$ and $\theta$ are the same angle.

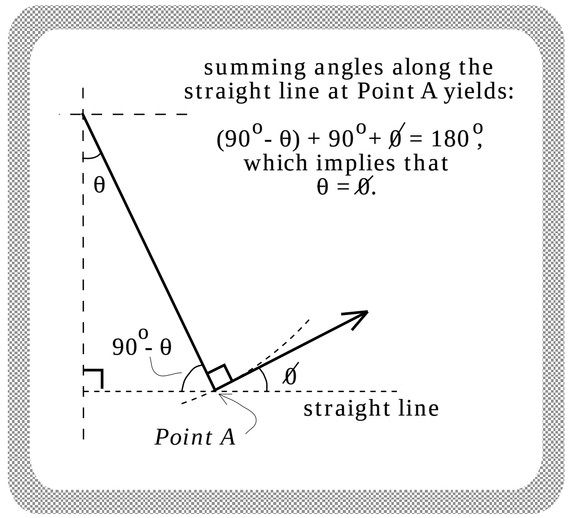


FIGURE 6.7c

**g.)** We are now ready to use our definition of *work* to determine the amount of work ***F*** does as the body moves from $\theta = 0^o$ to $\theta = \theta_1$. Specifically:

$$\begin{aligned}
W &= \int dW \\
&= \int \mathbf{F} \bullet d\mathbf{r} \\
&= \int \quad |\mathbf{F}| \quad\quad |d\mathbf{r}| \quad \cos\phi \\
&= \int [mg(\tan\theta)][L d\theta](\cos\theta) \\
&= mgL \int \left[\frac{\sin\theta}{\cos\theta}\right](\cos\theta) d\theta \\
&= mgL \int_{\theta=0}^{\theta_1} \sin\theta\, d\theta \\
&= mgL\left[-\cos\theta\right]_{\theta=0}^{\theta_1} \\
&= mgL\left[(-\cos\theta_1) - (-\cos 0^\circ)\right] \\
&= mgL\left[1 - \cos\theta_1\right]
\end{aligned}$$

**4.)** Using a *UNIT VECTOR* approach:

**a.)** For a *dot product* using a *unit vector* approach, we need both the *force* and *displacement* as unit vectors.

**b.)** We know the force's magnitude is $mg(\tan\theta)$ and its direction is always in the $+\boldsymbol{i}$ direction. That is:

$$\mathbf{F} = [mg\,(\tan\theta)]\mathbf{i}.$$

**c.)** We also know the displacement is:

$$d\mathbf{r} = (dx)\mathbf{i} + (dy)\mathbf{j} + (dz)\mathbf{k}.$$

**d.)** The only problem: The variable $\tan\theta$ is not explicitly a function of $x$ and/or $y$ variables. That is easily remedied by examining the displacement sketched in Figure 6.7d. Using trig on the right triangle, we get:

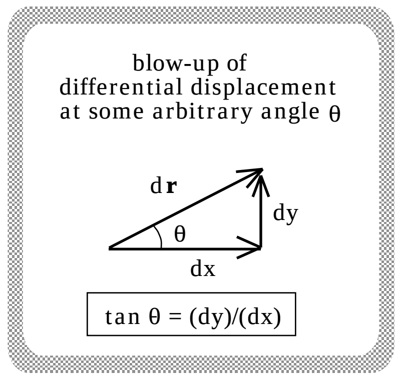


FIGURE 6.7d

$$\begin{aligned}\tan\theta &= (\text{opposite})/(\text{adjacent})\\ &= (dy)/(dx).\end{aligned}$$

**e.)** Putting it all together, we get:

$$\begin{aligned}
W &= \int dW\\
&= \int \mathbf{F}\bullet d\mathbf{r}\\
&= \int [mg(\tan\theta)\mathbf{i}]\bullet[(dx)\mathbf{i}+(dy)\mathbf{j}]\\
&= \int \left[mg\left(\frac{dy}{dx}\right)\mathbf{i}\right]\bullet[(dx)\mathbf{i}+(dy)\mathbf{j}]\\
&= \int \left[mg\left(\frac{dy}{dx}\right)dx\right]\\
&= \int_{y=0}^{y_{\theta_1}} (mg)\,dy\\
&= mg[y]_0^{y_{\theta_1}}\\
&= mgy_{\theta_1}
\end{aligned}$$

**f.)** Determining the *y coordinate* (call this $y_1$) of the ball when it is located at angle $\theta_1$ is not as difficult as it looks. Consider Figure 6.8. We know the string length is *L*. When in the vertical, the *string length* can also be written $L = L\cos\theta + y_1$. That means:

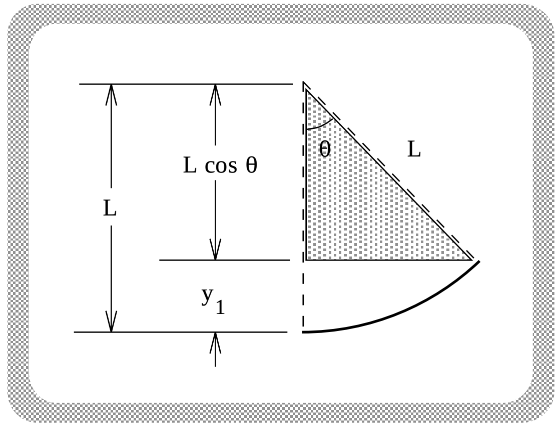

FIGURE 6.8

$$y_1 = L - L(\cos\theta).$$

**g.)** Using that relationship in our *work* expression, we get:

$$\begin{aligned} W &= mgy_1 \\ &= mg[L - L(\cos\theta)] \\ &= mg[L - L\cos\theta]. \end{aligned}$$

This is exactly the expression we determined using the *polar* approach on the problem.

**5.)** Don't be put off if a lot of the mathematical maneuvering you've just seen seems a bit mysterious. The reason you are in this class is to see how physicists use math to create theoretical models of the world. If physics was trivially obvious, you'd already know it all and there would be no reason to take the class!

## D.) The Work/Energy Theorem:

**1.)** We would like to relate the *total, net work* done on an object to its resulting change in *velocity*. This next section is the derivation of just such a relationship.

**Note 1:** You will not be held responsible for duplicating any of the material you are about to read in this part (i.e., *Part D-1*) except the *bottom line*. BUT, if you don't understand how we got there, the *bottom line* won't mean much. Also, the *work/energy theorem* is a half-way point to where we are really going. Understand it, but also understand that there is a more powerful presentation of the same idea coming up soon.

My suggestion is that you read this part, not for memorization purposes but for content. Follow each step as it comes *without* projecting ahead, and when you finally get to the end-result, take the time to reread the section to be sure you know *how* we got from start to finish.

**a.)** So far, we have been able to calculate the *work* $W_F$ a single force $\boldsymbol{F}$ does on a moving body. It isn't too hard to see that the total, *net work* $W_{net}$ due to all the forces acting on a body will equal the sum of the individual bits of work done by the individual forces.

What might not be so obvious is that there is another way to get that *net work* quantity. How so? We could determine the *net force* $\boldsymbol{F}_{net}$ acting on the body and use *it* in our *work* definition. Doing so yields:

$$W_{net} = \int \mathbf{F}_{net} \cdot d\mathbf{r}.$$

**b.)** By Newton's Second Law, the *net force* on an object will numerically equal the vector $m\boldsymbol{a} = m(d\boldsymbol{v}/dt)$. If, for ease of calculation, we assume that the *net force* and the *displacement* $d\boldsymbol{r}$ are both in the $\boldsymbol{i}$ *direction*, we can write the *dot product* associated with the *work* definition as:

$$\begin{aligned} W_{net} &= \int \mathbf{F} \bullet d\mathbf{r} \\ &= \int (m\mathbf{a}) \bullet d\mathbf{r} \\ &= \int \left[ m \frac{d(v)}{dt} \mathbf{i} \right] \bullet \left[ (dx)\mathbf{i} \right] \\ &= m \int \left( \frac{dv}{dt} \right) dx \qquad \text{(Equation A)} \end{aligned}$$

**c.)** As the velocity term is *time dependent* (otherwise, we wouldn't be able to determine *dv/dt*), we would like to write the displacement term *dx* in terms of time, also. To do so, note that the *rate at which the position changes with time* (i.e., *dx/dt*) times the *time interval dt* over which the change occurs, yields the *net change in position dx.* Put more succinctly:

$$dx = \left[ \left( \frac{dx}{dt} \right) (dt) \right].$$

**d.)** Substituting this into Equation A and manipulating as only physicists will do (i.e., after canceling out selected *dt* terms), we get:

$$W_{net} = m\int\left(\frac{dv}{dt}\right)\left[\left(\frac{dx}{dt}\right)dt\right]$$
$$= m\int dv\left[\frac{dx}{dt}\right].$$

**e.)** Noticing that *dx/dt* is the velocity $v$ of the body, and taking the limits to be from some velocity $v_1$ to a second velocity $v_2$, we can rewrite, then integrate this expression as:

$$W_{net} = m\int_{v_1}^{v_2}(v)dv$$
$$= m\left(\frac{v^2}{2}\right)_{v_1}^{v_2}$$
$$= \left(\frac{1}{2}\right)m(v_2)^2 - \left(\frac{1}{2}\right)m(v_1)^2.$$

**f.)** This equation, $W_{net} = (1/2)mv_2^2 - (1/2)mv_1^2$, is called the <u>*Work/Energy Theorem*</u>. It is the *bottom line* for this section.

**2.)** The quantity $(1/2)mv^2$ has been deemed important enough to be given a special name. It is called the *Kinetic Energy* of a body of mass $m$ moving with velocity $v$. Its units are (kg)(m/s)$^2$, or joules--the same units as *work* (as expected).

**a.)** OBSERVATION: Something is said to have *energy* if it has the ability to do *work* on another "something."

**i.)** Example--a car traveling at 30 m/s: A car has energy associated with its motion (i.e., kinetic energy). If this is not obvious, imagine stepping in front of one traveling down the road. Any damage done to you by the car will be due to the fact that the car has energy wrapped up in its motion and, as a consequence, has the ability to do work on you.

**ii.)** Example--a sound wave: If a sound wave didn't carry energy, it wouldn't have *the ability to do work* on the hairs in your ears which, when moved, produce the electrical signals your brain translates into sound.

**iii.)** In both of the cases cited above, *energy* is associated with *the ability to do work on something else.*

**b.)** Kinetic Energy--Example #1: What must the *magnitude* of the *velocity* of a 1000 kg car be if it is to have the same *kinetic energy* as a 2 gram bullet traveling at 300 m/s?

Solution:

$$
\begin{aligned}
KE_b &= (1/2)m_b v_b^2 \\
&= (1/2)\,(.002\text{ kg})\,(300\text{ m/s})^2 \\
&= 90\text{ joules.}
\end{aligned}
$$

If $KE_b = KE_c$:

$$(1/2) \quad m_c \quad v_c^2 \;= 90\text{ joules}$$

$$\Rightarrow \quad (1/2)\,(1000\text{ kg})\,(v_c)^2 = 90\text{ joules}$$

$$\Rightarrow \quad v_c = .42\text{ m/s.}$$

**c.)** Kinetic Energy--Example #2: If one triples a body's velocity, how does the body's *kinetic energy* change?

Solution:

$$
\begin{aligned}
KE_1 &= (1/2)mv_1^2 \\
KE_2 &= (1/2)m(3v_1)^2 \\
&= 9\,[(1/2)m(v_1)^2].
\end{aligned}
$$

As would be expected when the *kinetic energy* is proportional to the *square of the velocity*, tripling the speed increases the *kinetic energy* by a factor of three-squared, or nine.

**3.)** Back to the *Work/Energy Theorem*: whenever a *net amount of work* is done on a body, the body will either acquire or lose energy. That change will ALWAYS show itself as a change in the *kinetic energy* of the body.

More succinctly: the net work done on a body will always equal the *change of the body's kinetic energy*.

**4.)** Example: At a given instant, a 2 kg mass moving to the right over a frictional surface has a force $F = 5$ *nts* applied to the left at an angle $\theta = 30^o$ below the horizontal (see Figure 6.9). The

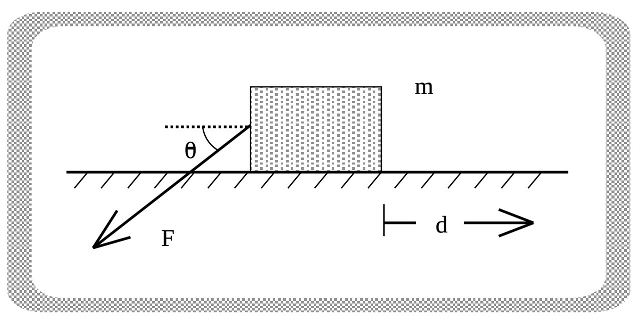


FIGURE 6.9

*average frictional force* acting on the box is $f_k = 1.5$ *nts*. If the block is initially moving with velocity 9 m/s, how fast will it be moving after traveling a distance 4 meters?

**Note:** You could have been given $\mu_k$ and been expected to use N.S.L. to determine the normal force $N$ required to use $f_k = \mu_k N$. That twist hasn't been included here for the sake of simplicity, but it is a perfectly legitimate problem for your next test.

**a.)** Someone well-familiar with the *work/energy theorem* would do the problem as shown below (if the pieces making up the expressions aren't self explanatory, a derivation of each follows in *Part b*):

$$W_{net} = \Delta KE$$

$$W_F \quad + \quad W_f \quad = \quad \Delta KE$$

$$(-Fd\cos\theta) \quad + \quad (-f_k d) \quad = \quad (1/2)mv_2^2 \quad - \quad (1/2)mv_1^2$$

$$-(5\text{ nts})(4\text{ m})(.866) - (1.5\text{ nts})(4\text{ m}) = (1/2)(2\text{ kg})(v_2)^2 - (1/2)(2\text{ kg})(9\text{ m/s})^2$$

$$\Rightarrow \quad v_2 = 7.59\text{ m/s}.$$

**b.)** The following shows how each quantity used in the above equations was derived:

**i.)** The *Work/Energy Theorem* states that:

$$W_{net} = \Delta KE.$$

**ii.)** The left-hand side of the equation is equal to the sum of all the work done by all the forces acting on the block. The f.b.d. shown in Figure 6.10 identifies those forces. Note that the work done by the *normal force* will always equal zero (the line of motion and the line of the *normal* are perpendicular to one another). The work due to *gravity* will, in this case, also equal zero for the same reason.

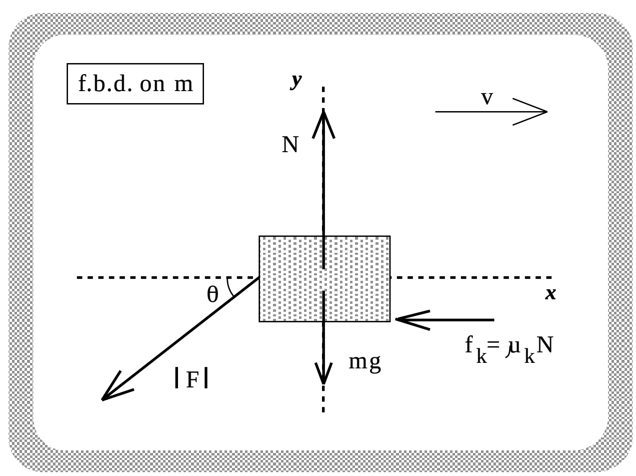


FIGURE 6.10

**iii.)** That leaves $W_{net} = W_F + W_{f_k}$.

**iv.)** Using the definition of work, we get:

$$\begin{aligned} W_F &= \mathbf{F} \cdot \mathbf{d} \\ &= |\mathbf{F}|\,|\mathbf{d}| \cos\phi \\ &= (F)\,(d) \cos(180^o - \theta) \\ &= -\,(F)\,(d) \cos\theta. \end{aligned}$$

and

$$\begin{aligned} W_{f_k} &= \mathbf{f_k} \cdot \mathbf{d} \\ &= |\mathbf{F}|\,|\mathbf{d}| \cos\phi \\ &= (f_k)\,(d) \cos 180^o \\ &= -\,f_k\, d. \end{aligned}$$

**Note 1:** The angle between *the line of motion* and the force ***F*** is not so obvious--we really did need to write out the *work* derivation for that force.

**Note 2:** Friction pulls energy out of the system, hence the negative work quantity. That energy is usually dissipated as *heat*.

**v.)** Putting it all together, we get:

$$\begin{aligned} W_{net} &= W_F + W_{f_k} \\ &= (-Fd\cos\theta) + (-f_k d). \end{aligned}$$

**vi.)** Returning to the Work/Energy theorem:

$$W_{net} = \Delta KE \qquad \text{(Equation A)}$$

$$\Rightarrow \qquad (-Fd\cos\theta) + (-f_k d) = (1/2)mv_2^{\,2} - (1/2)mv_1^{\,2} \qquad \text{(Equation B).}$$

**vii.)** We know everything except $v_2$. Solving for that variable, assuming $F = 5$ *newtons*, $f_k = 1.5$ *newtons*, $\theta = 30^o$, $d = 4$ *meters*, $m = 2$ *kg*, and $v_1 = 9$ *m/s*, we get:

$$(-Fd\cos\theta) + (-f_k d) = (1/2)\,mv_2^{\,2} - (1/2)mv_1^{\,2}$$
$$-(5\text{ nts})(4\text{ m})(.866) - (1.5\text{ nts})(4\text{ m}) = (1/2)\,(2\text{ kg})\,(v_2)^2 - (1/2)\,(2\text{ kg})\,(9\text{ m/s})^2$$
$$\Rightarrow \quad v_2 = 7.59\text{ m/s.}$$

**Note 1:** Do not memorize the final form of the above equation. The key is to understand *how we got it*. It is the approach that is important here, not the final result!

**Note 2:** Going back for another look at the original formulation of the problem (i.e., the way *you* ought to present a test problem should you be asked to use the *work/energy theorem* to solve a problem):

$$W_{net} = \Delta KE$$
$$W_F \quad + \quad W_f \quad = \quad \Delta KE$$
$$(-Fd\cos\theta) \quad + \quad (-f_k d) \quad = \quad (1/2)mv_2^{\,2} \quad - \quad (1/2)mv_1^{\,2}$$
$$-(5\text{ nts})(4\text{ m})(.866) - (1.5\text{ nts})(4\text{ m}) = (1/2)\,(2\text{ kg})\,(v_2)^2 - (1/2)\,(2\text{ kg})\,(9\text{ m/s})^2$$
$$\Rightarrow \quad v_2 = 7.59\text{ m/s.}$$

**Note 3:** For a moment, think about *the approach*. It allows you to relate the total amount of *energy-changing work* $W_{net}$ done on the body to the way the body's energy-of-motion (its *kinetic energy*) changes. Forces come into play in calculating the "work" part of the relationship. That means N.S.L. is still important (you could need it to determine an expression for the magnitude of an unknown force), but the main thrust is wrapped up in the question, "How does the system's ENERGY change?"

**Note 4:** Although the *Work/Energy Theorem* is important, we will shortly be using it to derive an even more important relationship. We haven't yet gotten to the "bottom line" of this approach.

**E.) Conservative Forces:**

**1.)** Background: A body of mass *m* moves from $y_1$ (call this Position 1) to $y_2$ (call this Position 2) with a constant velocity (see Figure 6.11). How much work does *gravity* do on the body as it executes the motion?

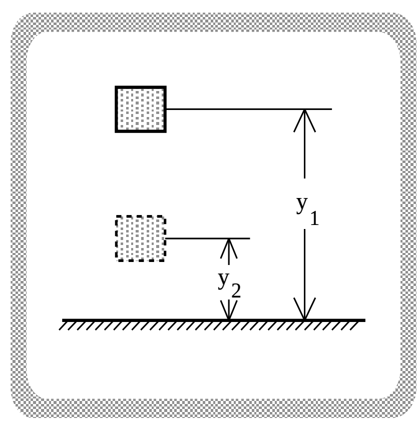

FIGURE 6.11

**Note:** There are at least two forces acting on the body in this case, one provided by gravity and one provided by an outside agent like yourself. Our only interest in this problem is in the work *gravity* does.

**a.)** Noting that the angle between the *line of the gravitational force* and the *line of the displacement vector* is $0^o$, we can use our definition of *work* to write:

$$
\begin{aligned}
W_{gr} &= \mathbf{F}_g \cdot \mathbf{d} \\
&= |\mathbf{F}|\,|\mathbf{d}| \cos\, 0^\circ \\
&= (mg)\,(y_1 - y_2)(1) \qquad \text{(Equation A),}
\end{aligned}
$$

which could be written:

$$
\begin{aligned}
&= -\,(mg)\,(y_2 - y_1) \\
&= -\,(mg)\,(\Delta y).
\end{aligned}
$$

**Note:** By definition, $\Delta y$ is the *final height* $y_2$ minus the *initial height* $y_1$. The last two steps of the above derivation were included to make use of this fact (this particular notation will come in handy later).

**2.)** Let us now replay the situation with a small alteration. Assume now that the block moves from *Point 1* to *Point 2* following the path outlined in Figure 6.12. How much work does *gravity* do on the block in this situation?

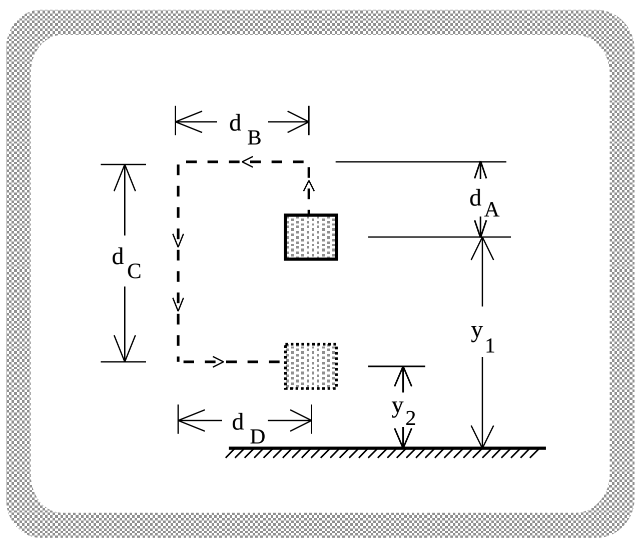


FIGURE 6.12

**a.)** Noting that the *total work* gravity does will equal the work done by gravity through each section of the displacement, we get:

$$W_{gr} = W_{d_A} + W_{d_B} + W_{d_C} + W_{d_D}.$$

**b.)** We know that the distance $d_C = d_A + (y_1 - y_2)$. Using that and the definition of *work*, we can write:

$$W_{gr} = (mg)(d_A)\cos 180^o + (mg)(d_B)\cos 90^o + (mg)(d_A + y_1 - y_2)\cos 0^o + (mg)(d_D)\cos 90^o.$$

**c.)** Setting *cos 180*$^o$*= -1*, *cos 90*$^o$*= 0*, and *cos 0*$^o$*= 1,* this becomes:

$$\begin{aligned} W_{gr} &= -(mg)(d_A) + (mg)(d_A + y_1 - y_2) \\ &= -(mg)(d_A) + (mg)(d_A) + (mg)(y_1 - y_2) \\ &= +(mg)(y_1 - y_2) \\ &= -(mg)(y_2 - y_1). \end{aligned}$$

**d.)** Notice that this is the same amount of work *gravity* did when the body followed the first path. In fact, no matter what path the body takes in moving from *Point 1* to *Point 2*, the amount of work gravity does on the body will *always be the same*.

Put another way, the *amount of work* gravity does on a body as the body moves from one point to another in the gravitational field is PATH INDEPENDENT. *FORCE FIELDS THAT ACT THIS WAY ARE CALLED* ***CONSERVATIVE FORCE FIELDS***.

**e.)** A corollary to this *path independence* observation is the fact that the amount of work a *conservative force field* does on a body that moves around a *closed path* in the field will always be ZERO!

**Note:** "Moving around a closed path" means the body ends up back where it started.

**i.)** Reasoning? Consider a body that moves upward a vertical distance *d*. The work *gravity* does on the body will be *-mgd* (negative because the angle between the *displacement vector* and the *gravitational force* is 180$^o$). When the body is brought back down to its original position, the work *gravity* does is *+mgd*. The *total work* gravity does on the body as it moves through the round trip is (*-mgd* + *mgd*), or ZERO.

Gravity is a *conservative force field*.

**f.)** An example of a force field that is not *conservative* is *friction*. Common sense dictates that the further a body moves under the influence of friction, the more work friction will do on the body. As an example, anyone who has ever dragged a fingernail across a chalkboard knows that the further one drags, the more *work* friction does on his fingernails (and the more his listening friends will want to murder him).

From another perspective, frictional forces always *oppose* the direction of relative motion between two bodies. This means that a frictional force will either do all *negative work* or all *positive work* (99% of the time it's negative), depending upon the situation. That, in turn, means that the work due to friction on a body moving around a closed path can never equal zero.

Friction is a *non-conservative force.*

**Note:** For those of you who are wondering if there are other kinds of non-conservative force fields, all *time-varying* force fields qualify. You will not be asked to deal with time-varying fields until much later; the only *non-conservative* force you will have to worry about for now is friction.

## F.) Preamble to the *Gravitational Potential Energy* Function:

**Note 1:** We are about to consider a concept you have heard about in past science classes but that was most probably never addressed in a truly rigorous way. To eliminate as much intellectual stress as possible, my suggestion is that you forget everything you have ever been told about *potential energy* and start from scratch with the presentation that follows.

**Note 2:** You will not be held responsible for duplicating any of the material you are about to read except *the bottom line.* BUT, if you don't understand the following material you *won't* understand the *bottom line*, and if you don't understand the "bottom line" you will undoubtedly find yourself totally lost later. Therefore, read the next section, not for memorization purposes but for content. Follow each step as it comes without projecting ahead. When you finally get to the end-result, read back over the material to be sure you know what assumptions were made in proceeding to the endpoint.

**1.)** Consider a conservative force field--gravity, for instance. A body of mass $m$ moves from $y_1$ (call this Position 1) to $y_2$ (call this Position 2) with a constant velocity. How much work does the *gravitational force field* do on the body as the body so moves?

**a.)** This was the question posed at the beginning of the "Conservative Forces" section. The solution was found to be:

$$W_{gr} = -(mg)(y_2 - y_1) \qquad \text{(Equation A).}$$

**b.)** One of the important conclusions drawn from that section was the observation that as a body moves from *Point 1* to *Point 2* in a gravitational field, the work done by the field is not dependent upon the path taken. Gravity is a *conservative force.*

**c.)** With that in mind, let's consider a novel idea. If the path counts for nothing--if the endpoints are all that are important when determining the work gravity does--might it not be possible to somehow define a number $N_1$ that can be attached to *Point 1*, and a number $N_2$ that can be attached to *Point 2*, and cleverly make them such that the *difference* between them would yield the amount of *work done by gravity* as the body proceeds from *Point 1* to *Point 2*?

**d.)** This surely is a strange idea, but whether you see the usefulness of it or not, could it be done?

The answer is "yes."

**e.)** Example: Figure 6.13 shows just such a possible situation. Assuming the numbers have been chosen appropriately, the work done on the body due to gravity as the body goes from *Points 1* to *2* should be:

$$\begin{aligned} W_{gr} &= (N_2 - N_1) \\ &= [(12 \text{ joules}) - (25 \text{ joules})] \\ &= -13 \text{ joules.} \end{aligned}$$

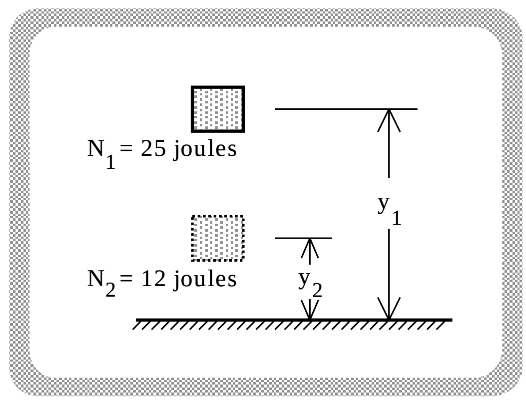

FIGURE 6.13

**f.)** There is only one difficulty with this. We have assigned *zero* to ground level making all numbers above ground level *increase* with elevation. That means that when a body moves from a *higher* (big number) position to a *lower* (small number) position, the difference between the second number and first number ($N_{low} - N_{hi}$) will be negative (just as we found in our example). The problem here is that if we proceed from high to low (i.e., move in the direction of *mg*), the work *gravity* does should be *positive*!

To make our scheme work, we need to modify our original model by re-defining the "numbers expression." We will do so by putting a *negative sign* in front of the relationship. This yields:

$$\begin{aligned} W_{gr} &= -(N_2 - N_1) \qquad \text{(Equation B)} \\ &= -[\,(12 \text{ joules}) - (25 \text{ joules})\,] \\ &= +13 \text{ joules.} \end{aligned}$$

**g.)** With our modification, we now have numbers attached to our initial and final points that, when correctly manipulated, give us the *work done by gravity* as the body moves from *Point 1* to *Point 2*.

**Note:** Kindly notice that we can do this only because the gravitational force is *conservative* and, hence, the work done due to gravity is *path independent*. If the work done depended upon the path taken, none of this would make any sense at all.

**h.)** It would be nice to have some handy mathematical function that would allow us to define our *N* numbers. Fortunately, we already have such a function for gravity. Using the definition of work, the *work done by gravity on a body moving from Point 1 to Point 2 in a gravitational field is*:

$$W_{gr} = -(mgy_2 - mgy_1).$$

We determined this expression earlier.

**i.)** By comparing this equation with our "number expression":

$$W_{gr} = -(N_2 - N_1),$$

we find by inspection that:

$$N_2 = mgy_2 \qquad \text{and} \qquad N_1 = mgy_1.$$

**j.)** Written in general (i.e., written as *mgy* where *y* is the <u>*vertical distance*</u> above some arbitrarily chosen *zero-height level*--the ground in our example), this function is important enough to be given a special name. It is called the "*gravitational potential energy*" function, normally characterized as $U_g$.

**k.)** <u>Bottom Line</u>: Although we have done this analysis using a gravitational force field, EVERY *conservative force field* has a *potential energy* function associated with it. Furthermore, there is a formal, Calculus-driven approach for deriving *potential energy* functions which will be presented shortly.

Whether you are given a potential energy function or have to derive it, understand that when a body moves through a *conservative force field* the amount of *work done by the field* as the body moves from *Point 1* to *Point 2* will always be:

$$W_{field} = -(U_{pt.2} - U_{pt.1}) \\ = -\Delta U.$$

This is the bottom line on *potential energy*.

## G.) Comments and Problems--Potential Energy Functions in General:

**1.)** Gravitational potential energy is not an absolute quantity.

**a.)** Consider the table and chalk shown in Figure 6.14. If we take *y* to be measured from the table's top (i.e., $y_1$ in the sketch), we are safe in saying that the amount of *potential energy* the chalk has is equal to $mgy_1$. If we want to determine the amount of work gravity does on the chalk as it rises to a second point at $y_2$, we can use the above-derived expression relating *gravitational potential energy* to *the work gravity does*, and get:

$$W_{gr} = -\Delta U_{gr} \\ = -(U_2 - U_1) \\ = -(mgy_2 - mgy_1).$$

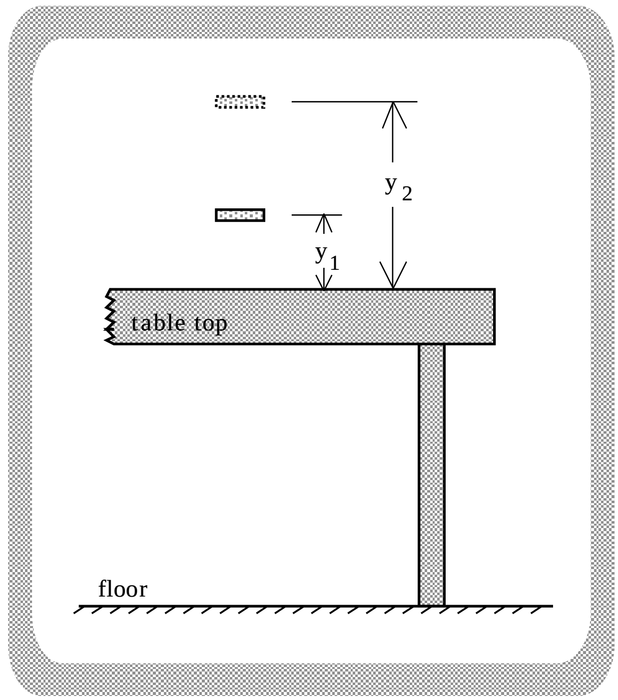

FIGURE 6.14

**b.)** Could we have used *the floor* as the *zero potential energy level*, making all *y* measurements from there?

ABSOLUTELY! The chalk would be assigned an initial *potential energy* value of $mgy_3$ (see Figure 6.15 on the next page), etc., and the work calculation would proceed as before:

$$W_{gr} = -\Delta U_{gr} \\ = -(U_4 - U_3) \\ = -(mgy_4 - mgy_3).$$

**c.)** The amount of work gravity does as the chalk rises to its new position can be determined correctly using *either* approach (notice that $y_2 - y_1$ is numerically equal to $y_4 - y_3$).

Why does this seemingly nonsensical situation exist? Because what is important is not the *amount* of gravitational potential energy an object has when at a particular point. What is important is the *change* of the gravitational potential energy of a body *as it moves from one point to another*. That is what allows us to determine the *amount of work* done on the body as it moves through the gravitational field. Thus, *work determination* is the *ONLY USE* you will ever have for *potential energy functions* . . . ever.

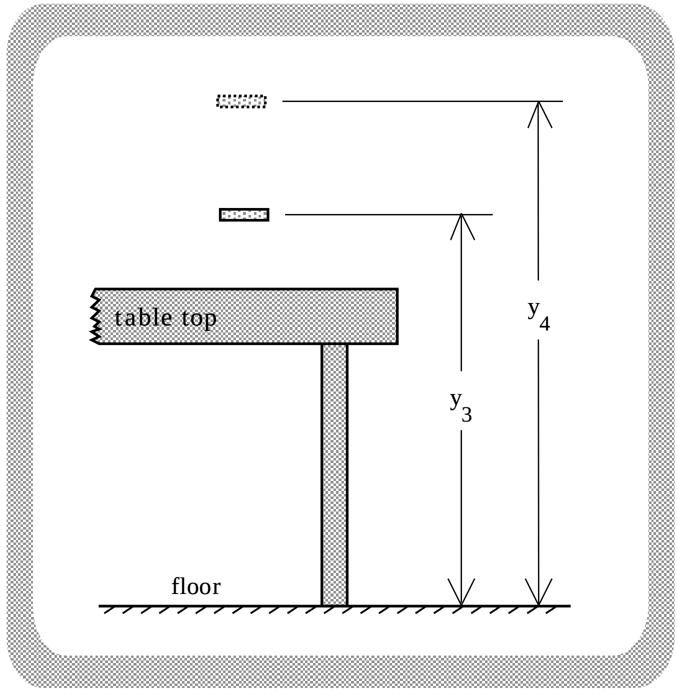


FIGURE 6.15

**2.)** Although most students associate *potential energy* with <u>*gravitational*</u> *potential energy*, there are actually many other *conservative force fields*. For instance, an ideal spring produces a force that is, at least theoretically, conservative. *All conservative forces have potential energy functions associated with them*.

Their use?

If you want to know *how much work* a conservative force field does on a body moving from one point to another within the field, and if you know the field's *potential energy function*, the work done by the field will always equal *minus the change of the potential energy function between the start and end points*, or:

$$W_{cons.force} = -\ \Delta U.$$

**3.)** <u>A Work/Energy-Theorem, Potential-Energy Example Problem:</u> A plane oriented at $30^o$ above the horizontal moves at 300 m/s. It is 1200 meters above the ground when a coke bottle becomes free and sails out of the window *a' la* the movie The Gods Must Be Crazy (see Figure 6.16). Neglecting air friction, how fast will the bottle be moving just before it hits the ground?

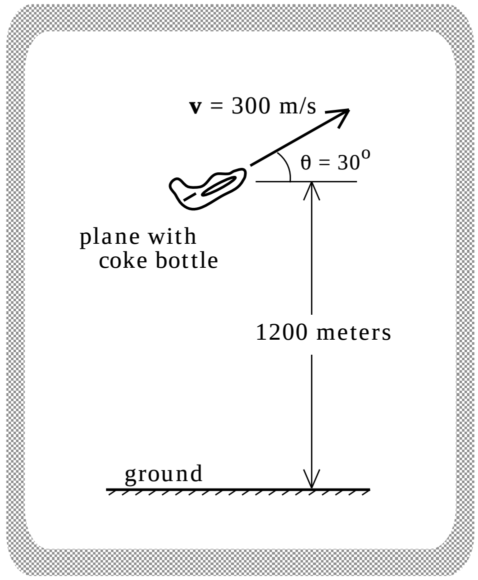


FIGURE 6.16

**a.)** The work/energy theorem states that the *net work* done on a body must equal the body's change in *kinetic energy* ($\Delta KE$). Mathematically, this is stated as:

$$W_{net} = \Delta KE.$$

**b.)** In this case, the $W_{net}$ consists solely of the *work done by gravity* $W_g$. Coupling this with the fact that there is a change in the *kinetic energy* $\Delta KE = (1/2)mv_2^2 - (1/2)mv_1^2$, we can write:

$$W_g = (1/2)mv_2^2 - (1/2)mv_1^2.$$

**c.)** If we had to calculate the work due to gravity using only the definition, the task would require Calculus (the bottle's *direction of motion* is constantly changing, which means the *angle* between the gravitational force and the displacement is constantly changing--see Figure 6.17) which would be nasty. Fortunately for us, we can easily determine the work gravity does in this situation because:

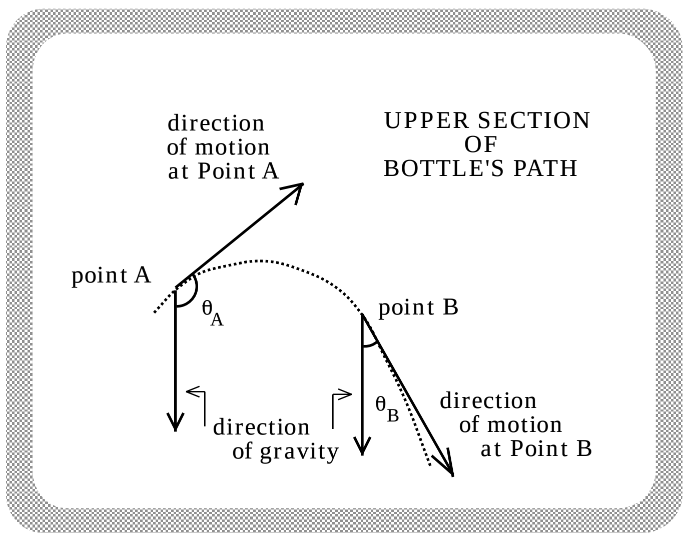


FIGURE 6.17

**i.)** We know the *potential energy function* for gravity is *mgy*; and

**ii.)** We know that:

$$W_g = -\Delta U_g = -(U_{2,g} - U_{1,g}).$$

**d.)** Utilizing these facts, we find:

$$W_g = (1/2)mv_2^2 - (1/2)mv_1^2$$
$$-(U_2 - U_1) = (1/2)mv_2^2 - (1/2)mv_1^2$$
$$-(mgh_2 - mgh_1) = (1/2)mv_2^2 - (1/2)mv_1^2.$$

**e.)** Solving for $v_2$ yields:

$$v_2 = [v_1^2 - 2(gh_2 - gh_1)]^{1/2}.$$

**f.)** Putting in the numbers yields:

$$v_2 = [(300 \text{ m/s})^2 - 2[(9.8 \text{ m/s}^2)(0 \text{ m}) - (9.8 \text{ m/s}^2)(1200 \text{ m})]]^{1/2}$$
$$= 336.9 \text{ m/s}.$$

**Note:** As usual, memorizing this result is a waste of time. What is important is the technique involved. Whenever you need to know *how much work a conservative force does* on a body moving through its force field, that quantity will always equal $-\Delta U$, where $U$ is the *potential energy function* associated with the field.

## H.) Deriving the *Potential Energy Function* for a Known Force Field:

**1.)** We created the idea of a *potential energy function* out of the need to easily determine the amount of work gravity does as a body moves from one point to another in a gravitational field. We then concluded that *any* conservative force can have a *potential energy function* associated with it. The only requirement? That:

$$W_{cons.fld.} = -\Delta U \qquad \text{(Equation A)},$$

where the symbol $U$ was used to denote the *potential energy function* associated with the conservative force field with which we happen to be dealing.

**2.)** It is possible to use the above expression to design an approach by which the *potential energy function* of a known, conservative force field can be derived. Specifically:

**a.)** In general, the work done by any force is:

$$W = \int dW = \int \mathbf{F}\cdot d\mathbf{r} \qquad \text{(Equation B).}$$

**b.)** Putting our two *work* expressions together (i.e., Equation A and Equation B), we get:

$$\begin{aligned} W_{cons.fld.} &= -\Delta U \\ &= \int \mathbf{F}_{cons.force} \cdot d\mathbf{r}. \end{aligned}$$

More succinctly, if a body is moving in the *x direction* from coordinate $x_1$ to coordinate $x_2$, we can write:

$$[U(x_2) - U(x_1)] = -\int_{x_1}^{x_2} \mathbf{F}_{cons.force} \bullet d\mathbf{r},$$

where the *differential displacement* is $d\mathbf{r} = dx\mathbf{i}$ and the integral must be evaluated (as shown) between $x_1$ and $x_2$.

**Note:** In advanced physics books, this relationship is expressed in a more general expression:

$$[U(\mathbf{r}_2) - U(\mathbf{r}_1)] = -\int \mathbf{F}_{cons.fld}\cdot d\mathbf{r},$$

where the *differential displacement* is $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$ and the integral must be evaluated between the generalized, two or three dimensional coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$. Also, the symbol used for *potential energy* in some books is *PE* instead of *U*.

**c.)** How does this help us define a *potential energy function*? Being clever, we can set $x_1$ equal to the coordinate at which the *potential energy* is to equal zero, then set $x_2$ to any arbitrary coordinate $x$. In that way, we end up with an integral we can do (we know $\mathbf{F}$) that has limits we can evaluate, and that is equal to $U(x) - 0$ or, simply, $U(x)$.

That is exactly what we want--a function that is general and that reflects the *potential energy of the system* at any coordinate $x$.

**3.)** An example: The gravitational force field close to the earth's surface applies a force equal to $-mg\mathbf{j}$ *newtons* to any body placed in the field. What is the *potential energy function* for this *force field*?

**a.)** The first thing to decide is where the *potential energy function* is to equal zero. IN MOST CASES, THE *POTENTIAL ENERGY FUNCTION* IS DEFINED AS ZERO WHERE THE *FORCE FUNCTION,* ITSELF, EQUALS ZERO. The only cases in which this isn't true are cases in which the force is a constant (hence, the force *has* no place where it is zero). Gravity is one such force. That means there is no preferred *zero potential energy level* for gravity close to the earth's surface. As such, we will arbitrarily choose *ground level* (i.e., $y = 0$) to be the *zero level* and work from there.

**b.)** Defining $U(y = 0) = 0$, we can use our definition of *potential energy* to write:

$$[U(y) - U(y=0)] = -\int \mathbf{F} \bullet d\mathbf{r}$$

$$\Rightarrow \quad U(y) - 0 = -\int_{y=0}^{y} [-mg\mathbf{j}] \bullet [dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}]$$

$$\Rightarrow \quad U(y) = \int_{y=0}^{y} (mg)dy$$

$$= mgy\Big|_{y=0}^{y}$$

$$= mgy$$

**Note:** $U(y = 0)$ is ZERO, not because $y = 0$ but because we have chosen to start our *work* calculation at the place where the *potential energy* has been defined as ZERO. In this case, that just happens to be at the coordinate $y = 0$.

**c.)** This is exactly the *potential energy function* we determined using the hand-waving arguments employed in the previous section.

**4.)** A more challenging example: The gravitational force field between *any two bodies* varies depending upon how massive the bodies are and how far their *centers of mass* are from one another. Newton deduced this general force function for gravity as:

$$\mathbf{F} = [-G(m_1\ m_2)/r^2]\mathbf{r},$$

where $G$ is called the *universal gravitational constant*, $m_1$ and $m_2$ are the masses in question, $r$ is the distance between their mass centers, and $\boldsymbol{r}$ is a unit vector in the *radial* direction (gravitational forces are always directed along a line between the two bodies--i.e., in a *radial* direction). The question? What is the *potential energy function* for this *force field?*

**a.)** For simplicity, assume the gravitational force is pointed in the *x direction*, making

$$\mathbf{F} = [-G(m_1\ m_2)/x^2]\mathbf{i}.$$

**b.)** Having made that simplification, the first thing to decide is where the *potential energy function* is equal to zero. Using the criterion suggested previously, the *potential energy* should be zero where the *force function* is equal to zero. That occurs at $x = \infty$.

**Note:** Remember what we are essentially doing when we use this approach? We are calculating the amount of work our force field does (i.e., $\int \boldsymbol{F}\cdot d\boldsymbol{r}$) as we move from the *zero potential energy point* to an arbitrary point within the field. In this case, we are moving from $x = \infty$ to some point at coordinate $x$.

**c.)** Defining $U(x = \infty) = 0$, we can use our definition of *potential energy* to write:

$$\begin{aligned}
\left[U(x) - U(x=\infty)\right] &= -\int \mathbf{F} \bullet d\mathbf{r} \\
\Rightarrow \quad \left[U(x) - 0\right] &= -\int \left[-G\frac{m_1 m_2}{x^2}\,\mathbf{i}\right] \bullet \left[dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}\right] \\
&= -\int_{x=\infty}^{x} \left[-G\frac{m_1 m_2}{x^2}\right] dx \\
&= \left(Gm_1 m_2\right)\left[\frac{-1}{x}\right]_{x=\infty}^{x} \\
&= \left(Gm_1 m_2\right)\left[\left(\frac{-1}{x}\right) - \left(\frac{-1}{\infty}\right)\right] \\
&= -G\frac{m_1 m_2}{x}
\end{aligned}$$

**d.)** This is the *potential energy function* for gravitational fields anywhere. Does it work? Let's see. According to the theory, we should be able to calculate the amount of work gravity does as a body moves from one point to another in a conservative force field using:

$$W = -\Delta U.$$

Assume your mass is 85 kg. You're in an elevator moving upward from ground level to a position 200 meters above the ground. How much work does *gravity* do as you so move?

**i.)** Using the *gravitational potential energy function* we derived for situations near the surface of the earth (i.e., $U_{mg,near} = mgy$, where we can assume ground level is the *zero potential energy level*), the amount of work done by gravity is found to be:

$$
\begin{aligned}
W_{grav} &= -[\quad U(y_2 = 200) \quad - U(y_1 = 0)] \\
&= -[\quad mgy_2 \quad - \quad mgy_1 \quad ] \\
&= -[(85\text{ kg})(9.8\text{ m/s}^2)(200\text{ m}) - \quad 0 \quad ] \\
&= -166{,}600\text{ joules.}
\end{aligned}
$$

**ii.)** We would like to do the same problem using the <u>*general*</u> *potential energy function* for gravity (i.e., $-Gm_1m_2/r^2$, where $r$ is the distance between the center of masses of the interacting objects--in this case, you and the earth). To do so, note that:

--the universal gravitational constant $G = 6.67\text{x}10^{-11}\ m^3/kg{\cdot}s^2$;

--the mass of the earth is $m_e = 5.98\text{x}10^{24}\ kg$;

--the radius of the earth is $r_e = 6.37\text{x}10^{6}\ m$.

**iii.)** Remember how *potential energy functions* are used. If you want the amount of work done by a conservative field as a body moves from one point to another in the field, evaluate the *potential energy function* for the field AT THE START POINT and AT THE END POINT, then take *minus* the difference of that amount. The value you end up with will be the work done by the field during the motion. Up until now, all you have worked with has been the simple version of *gravitational potential energy*--a function with an adjustable *zero level*. You are about to use a *potential energy function* with a fixed *zero point* (remember, $U = 0$ at <u>*infinity*</u> for this function). Even though you may be in the habit of treating ground level as the *zero point*, that isn't true of this function! With that in mind:

**iv.)** If we let $r_e$ be the distance between the earth's center and your center of mass when standing on the earth's surface (this is essentially the radius of the earth), then $r_e + 200$ will be the distance between your *center of mass* when at 200 meters above the earth's surface. We can write:

$$W_{grav} = -\left[\quad U(r_e + 200) \quad - \quad U(r_e) \quad\right]$$
$$= -\Big[[-Gm_e m_{you}/(r_e + 200)] - [-Gm_e m_{you}/(r_e)]\Big].$$

**v.)** Pulling out the constants, eliminating the units for the sake of space, and wholly ignoring significant figures, this becomes:

$$\begin{aligned} W_{grav} &= G \quad m_e \quad m_{you}[1/(r_e + 200) - 1/ \quad (r_e) \quad ] \\ &= (6.67\text{x}10^{-11})\ (5.98\text{x}10^{24})\ (85)\ [1/(6{,}370{,}200) - 1/(6{,}370{,}000)] \\ &= 5322220652 - 5322387755 \\ &= -167{,}103 \text{ joules.} \end{aligned}$$

**vi.)** Using the *near Earth potential energy function* in *Part d-i* above, we found that gravity did -166,600 joules of work. If we had not used rounded values for *G, $r_e$*, and *$m_e$*, these two numbers would have been the same.

**vii.)** <u>Bottom Line</u>: Our approach for determining *potential energy functions* generates functions that work as expected. LEARN THE APPROACH!

## I.) The Potential Energy Function for an Ideal Spring:

**1.)** An ideal spring loses no energy as it oscillates back and forth. The amount of work such springs do through one full cycle is zero, which is to say that the force they provide is a conservative one. As such, we can derive a *potential energy function* for an ideal spring using the approach outlined above.

**2.)** The position of a body attached to an ideal spring is measured from the system's equilibrium position (i.e., the position at which the force on the body *due to the spring* is zero). It has been experimentally observed that when a mass is attached to a spring and the spring is elongated or compressed:

**a.)** The magnitude of the *spring force* exerted on the body is proportional to the spring's displacement from the equilibrium, and

**b.)** The direction of the force always points toward the equilibrium position.

**c.)** Assuming the force is in the *x direction,* these observations can be mathematically expressed as:

$$\mathbf{F} = -kx\mathbf{i},$$

where *k* is a constant that defines the amount of force required to compress the spring *one meter*, and *x* is the distance the spring is displaced from its equilibrium position (see Figure 6.18 to the right).

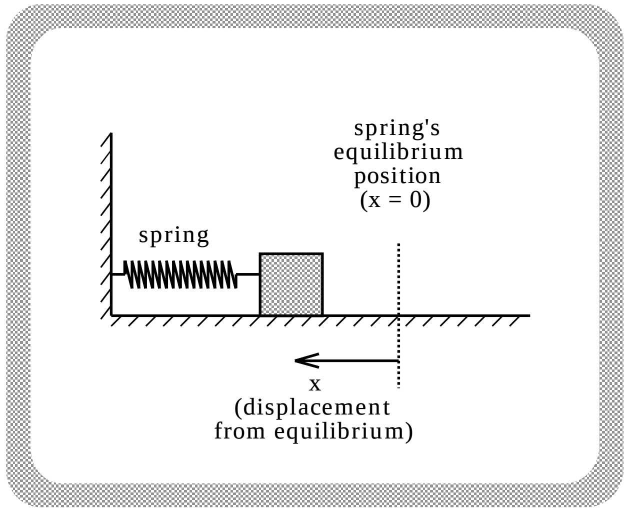


FIGURE 6.18

**Note:** The displacement term *x* is really a $\Delta x$, but as usual the convention is to assume that the initial position is at $x = 0$. This leaves the displacement term as $\Delta x = x - 0 = x$.

**3.)** Noting that the force function is ZERO at $x = 0$ (hence, the *potential energy function* must be defined as *ZERO at* $x = 0$), we can write:

$$\begin{aligned}
\left[U(x) - U(x=0)\right] &= -\int_{x=0}^{x} \mathbf{F}_{spr} \bullet d\mathbf{r} \\
\Rightarrow \qquad U(x) &= -\int_{x=0}^{x} \left[-kx\mathbf{i}\right] \bullet \left[dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}\right] \\
&= \int_{x=0}^{x} (kx)dx \\
&= k\left[\frac{x^2}{2}\right]_{x=0}^{x} \\
&= \left(\frac{1}{2}\right)kx^2
\end{aligned}$$

**4.)** Having derived the *potential energy function* for an ideal spring as

$$U_{sp} = (1/2)k(x)^2,$$

we can now use that function in any problem in which an ideal spring does work.

**5.)** <u>A Problem Involving a *U Function* Other Than Gravity (i.e., that of a Spring):</u> A 2 kilogram block on a horizontal surface is placed <u>without attachment</u> against a spring whose spring constant is $k = 12\ nt/m$. The block is made to compress the spring .5 meters (see Figure 6.19 below). Once done, the block is released and accelerates out away from the spring. If it slides over 2 meters of *frictionless surface* before sliding onto a *frictional* surface, and if it then proceeds to travel an *additional* 13 meters on the *frictional surface* before coming to rest, how large was the *frictional force* that brought it to rest?

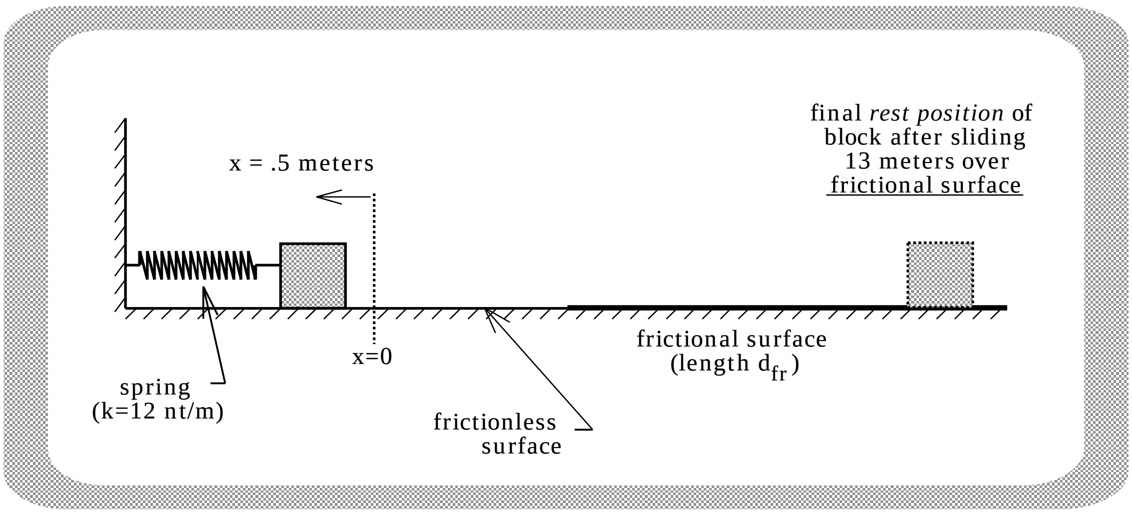


FIGURE 6.19

**Note:** Do not get too comfortable with using the *work/energy theorem*. It is an OK approach in some cases, but there is a much easier way to deal with the kind of information given in this problem using the concept of *energy conservation*. That alternative approach will be presented shortly. This example is given SOLELY to allow you to see a *potential energy function* other than *gravity* in a problem.

**a.)** Looking at this problem from a work/energy perspective, we need to determine two different quantities: *the net change of the body's kinetic energy* (i.e., its final *kinetic energy* minus its initial *kinetic energy*), and *the amount of work done by all forces acting on the body* between the beginning and end of its motion. In short, we need to determine:

$$W_{net} = \Delta KE.$$

**b.)** As the mass does not *rise* or *fall* in this problem, gravity does no work and there is no reason to include the *potential energy function* for gravity in the *work/energy* expression.

**c.)** Writing this out as you would on a test (should you be asked to use the *work/energy theorem* on a test), we get:

$$W_{net} = \Delta KE$$

$$\Rightarrow \quad W_{sp} \quad + \quad W_{fr} \quad = \quad KE_2 \quad - \quad KE_1$$

$$-\Delta U_{sp} \quad + \quad (-f_k d_{fr}) \quad = (1/2)\, m\, {v_2}^2 - (1/2)\, m\, {v_1}^2$$

$$-[0 - (1/2)kx^2] \quad + \quad (-f_k d_{fr}) \quad = (1/2)\, m\, {v_2}^2 - (1/2)\, m\, {v_1}^2$$

$$.5(12\ nt/m)(.5\ m)^2 + (-f_k)(13\ m) = .5(2\ kg)(0)^2 - .5(2\ kg)(0)^2$$

$$\Rightarrow \quad f_k = .115\ nts.$$

**Note:** Once again, THE WORK DONE BY A *CONSERVATIVE FORCE FIELD* ON A BODY MOVING THROUGH THE FIELD WILL ALWAYS EQUAL *$-(U_2 - U_1)$*, ASSUMING THE *POTENTIAL ENERGY FUNCTION* USED IS THE PROPER FUNCTION FOR THE FORCE FIELD.

## J.) Force Derived from Known Potential Energy Function:

**1.)** We have already established an approach for deriving the *potential energy function* associated with a given *force function*. How might we go the other way (i.e., derive a *force function* for a given *potential energy function*)?

**2.)** In one dimension:

**a.)** We can write:

$$\int_{x_1}^{x_2} d(U) = \left[U(x_2) - U(x_1)\right] = -\int_{x_1}^{x_2} \mathbf{F} \bullet d\mathbf{x},$$

where the first integral is equal to the middle expression by definition, and the second expression equals the third due to the derivation of the potential energy function.

**b.)** This implies that:

$$d(U) = -\mathbf{F} \bullet d\mathbf{x},$$

which, in turn, implies that:

$$F_x = -\frac{d(U)}{dx}.$$

**3.)** It would be useful to have a way to denote this operation in its most general sense so that we can determine the magnitude of components of the force function along with appropriate unit vectors. That possibility exists using partial derivatives and the *del operator*. Specifically:

**a.)** As:

$$\nabla(U) = \left[\frac{\partial(U)}{\partial x}\mathbf{i} + \frac{\partial(U)}{\partial y}\mathbf{j} + \frac{\partial(U)}{\partial z}\mathbf{k}\right]$$
$$= -\left[F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}\right].$$

**b.)** We can write:

$$\mathbf{F} = -\nabla(U).$$

**Note:** Interesting observation: This means that the *rate of change* of a *potential energy function* in a particular direction numerically equals the *force component in that direction* (that is, the force component of the force function that is associated with the potential energy function).

## K.) MODIFIED CONSERVATION OF ENERGY Theorem: Or, Getting to the Bottom of the Bottom Line:

**Note:** We are about to put the work/energy theorem into a considerably more useful form. To do so, we will spend some time with the derivation behind "the bottom line." You will not be asked to duplicate this derivation, but if you do not understand it, you will most probably not be able to use the end result to its full extent. Read the following section; think about it; then read it again. It is important that you know what is being done here.

**1.)** Consider an object with numerous forces acting on it as it moves from *Point A* to *Point B*. The *work/energy theorem* relates the *amount of work done* on the body to the body's *change of kinetic energy*. Writing this out, we get:

$$W_{net} = \Delta KE.$$

The left-hand side of this equation is simply the sum of the work done by all the forces acting on the body. This equation could be written as:

$$W_A + W_B + W_C + W_D + \ldots = \Delta KE,$$

where $W_A$ is the work done by force $\boldsymbol{F}_A$, $W_B$ is the work done by force $\boldsymbol{F}_B$, etc. For the sake of argument:

**a.)** Assume forces $\boldsymbol{F}_A$ and $\boldsymbol{F}_B$ are *conservative forces* with *known potential energy* functions $U_A$ and $U_B$. If we define the body's *potential energy* when at *Point 1* due to force field $\boldsymbol{F}_A$ as $U_{A,1}$, and the *potential energy* when at *Point 2* due to force field $\boldsymbol{F}_A$ as $U_{A,2}$, then the *work done* by $\boldsymbol{F}_A$ as the body moves from *Point 1* to *Point 2* in the force field will be:

$$\begin{aligned} W_A &= -\Delta U_A \\ &= -(U_{A,2} - U_{A,1}). \end{aligned}$$

Likewise, the *work done* on the body due to $\boldsymbol{F}_B$ will be:

$$\begin{aligned} W_B &= -\Delta U_B \\ &= -(U_{B,2} - U_{B,1}). \end{aligned}$$

**b.)** Assume the forces associated with $W_C$ and $W_D$ are either *non-conservative forces* that have no *potential energy function* or *conservative forces* for which we don't know the *potential energy function.* If that be the case, we will have to determine those *work* quantities using:

$$W_C = \mathbf{F}_C \cdot \mathbf{d}$$

and

$$W_D = \mathbf{F}_D \cdot \mathbf{d}.$$

**c.)** Having made these assumptions, we can write the *work/energy theorem* as:

$$W_A + W_B + W_C + W_D + \ldots = \Delta KE,$$

or

$$[-(U_{A,2} - U_{A,1})] + [-(U_{B,2} - U_{B,1})] + (\mathbf{F}_C \cdot \mathbf{d}) + (\mathbf{F}_D \cdot \mathbf{d}) + \ldots = (1/2)mv_2^{\,2} - (1/2)mv_1^{\,2}.$$

**d.)** Multiplying the *potential energy quantities* by the *-1* outside their parentheses, we get:

$$(-U_{A,2} + U_{A,1}) + (-U_{B,2} + U_{B,1}) + (\mathbf{F}_C \cdot \mathbf{d}) + (\mathbf{F}_D \cdot \mathbf{d}) + \ldots = (1/2)mv_2^{\,2} - (1/2)mv_1^{\,2}.$$

**e.)** The expression we end up with has:

**i.)** A number of *potential energy functions* evaluated at $t_1$ (i.e., when the body is at *Point 1);*

**ii.)** A number of *potential energy functions* evaluated at $t_2$ (i.e., when the body is at *Point 2*);

**iii.)** The *kinetic energy function* evaluated at $t_1$;

**iv.)** The *kinetic energy function* evaluated at $t_2$;

**v.)** And all the other work done on the body that we haven't been able to keep track of using *potential energy functions*, but that has been done on the body as it moved from *Point 1* to *Point 2*.

**f.)** If we move all the *time 1* terms to the left-hand side of the equation and all the *time 2* terms to the right-hand side, our equation will look like:

$$(1/2)mv_1^{\,2} + U_{A,1} + U_{B,1} + (\mathbf{F}_C \cdot \mathbf{d}) + (\mathbf{F}_D \cdot \mathbf{d}) + \ldots = (1/2)mv_2^{\,2} + U_{B,2} + U_{A,2}.$$

**g.)** What we have now is the *kinetic energy* of the body at *Point 1* added to the sum of the *potential energies* attributed to the body while at *Point 1* added to all the extraneous work done on the body (extraneous in the sense that we haven't kept track of it with potential energy functions) between *Points 1* and *2* equaling the *kinetic energy* of the body when at *Point 2* added to the sum of the *potential energies* of the body while at *Point 2*.

Written in shorthand, this is:

$$KE_1 + \Sigma U_1 + \Sigma W_{extraneous} = KE_2 + \Sigma U_2.$$

**h.)** This is called the *modified conservation of energy* equation. If we identify the sum of the kinetic and potential energies of a body while at a particular point (that is, $KE_1 + \Sigma U_1$) as "the total mechanical energy $E_1$" of the body at that point in time, the *modified conservation of energy* equation can be written in an even more compact way:

$$E_1 + \Sigma W_{extraneous} = E_2.$$

In this form, the equation states that the *total energy* of the body when at *Point 1* will equal the *total energy* of the body when at *Point 2*, modified only by the "extraneous work" done to the body as it moves *from Points 1 to 2*. In other words, this equation keeps track of the ENERGY the body either *has* or *has-the-potential-of-picking-up* as it moves from one point to another.

**Note:** The word "conserved" here means "not changing with time." If we have no extraneous bits of work being done as the body moves from *Point 1* to *Point 2*, which is to say we know the *potential energy functions* for all the forces doing work on the body as it moves and there are no non-conservative forces acting on the system, we can write $E_1 = E_2$. This is the true "conservation of energy" equation. That equation is the mathematical way of saying, "The *total energy* of the system will always be the same--the body's *kinetic energy* may change and its *potential energy* may change, but the *sum* of the *kinetic* and *potential energies* will be a constant throughout time."

By adding the possibility of dealing with non-conservative or oddball conservative forces (one for which we haven't a potential energy function), the "modified" *conservation of energy* equation is extremely powerful. It allows for the analysis of situations in which $E_1$ and $E_2$ are *not* equal but are related in a deducible way.

**2.)** Bottom Line: When approaching a problem from the standpoint of energy considerations:

**a.)** Determine the amount of *kinetic energy* the body has to start with (this may be nothing more than writing down $(1/2)mv_1^2$) and place that information on your sketch next to the body's position at *Point 1*. Do the same for *Point 2*.

**b.)** Identify any conservative forces for which you know *potential energy functions*. Once identified, determine the *amount of potential energy* the body has when at *Point 1* and put that information on your sketch. Do the same for *Point 2*.

**Note:** If *gravity* is the only force with *potential energy function* in the problem, this last step may amount to nothing more than writing $U_1 = mgh_1$ next to *Position 1* on your sketch with a similar notation at *Position 2*.

**c.)** Identify any forces that do work on the body as it moves from *Point 1* to *Point 2*, but for which you don't have *potential energy functions*. Determine the *amount of work* they do over the motion and place that information in a convenient spot on your sketch.

**d.)** Take the information gleaned from *Parts a, b,* and *c,* and after writing out $KE_1 + \Sigma\, U_1 + \Sigma\, W_{extraneous} = KE_2 + \ \Sigma\, U_2$, plug the information in where appropriate. Solve for the unknown(s) in which you are interested.

**3.)** <u>A Simple Example:</u> Consider a ball of mass .25 kilograms positioned at $y_1 = +4$ *meters* above the ground. It is given an initial upward velocity of 6 m/s at a $60^o$ angle with the horizontal. The ball freefalls, finally reaching $y_2 = 1$ *meter* above the ground. If friction does 7 joules of work on the ball during the trip, how fast is the ball moving when it gets to $y_2 = 1$ *meter*?

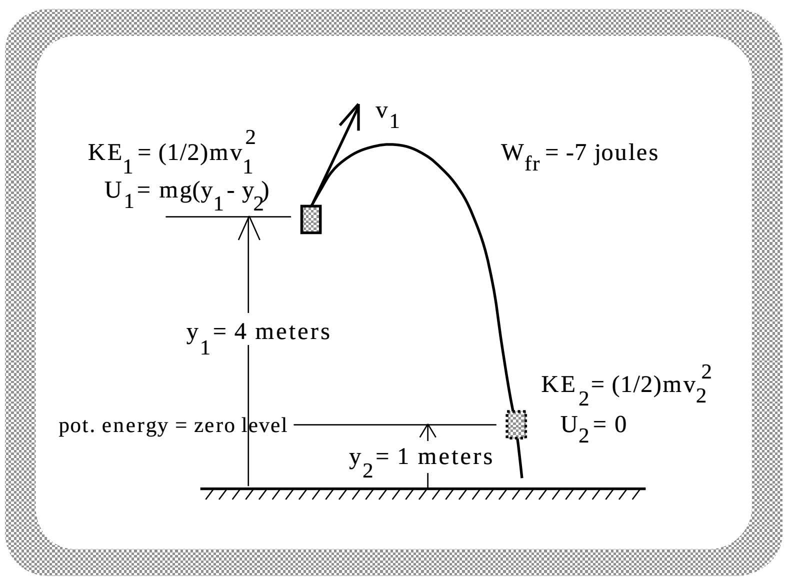


FIGURE 6.20

**a.)** Consider the sketch in Figure 6.20. In it is placed all the information needed to solve this problem. WE WILL ASSUME THE *ZERO POTENTIAL ENERGY* LEVEL is AT THE "FINAL POSITION" (i.e., $y_2$).

**b.)** Remembering that the *work due to friction* is negative and that the *zero potential energy level* is at $y_2$, we can begin with the *modified conservation of energy equation* and write:

$$KE_1 + \Sigma\, U_1 + \Sigma\, W_{extraneous} = KE_2 + \Sigma\, U_2.$$

**c.)** Spreading out that equation to see what goes where, then solving, we get:

$$
\begin{array}{ccccccccc}
KE_1 & + & \Sigma\, U_1 & + \Sigma\, W_{ext} & = & KE_2 & + & \Sigma\, U_2 \\
(1/2)mv_1^{\,2} & + & mg(y_1 - y_2) & + \quad W_{fr} & = & (1/2)mv_2^{\,2} & + & 0 \\
.5(.25\text{ kg})(6\text{ m/s})^2 & + & (.25\text{ kg})(9.8\text{ m/s}^2)[(4\text{ m}) - (1\text{ m})] & + (-7\text{ j}) & = & .5(.25\text{ kg})v_2^{\,2} & + & 0
\end{array}
$$

$$\Rightarrow \quad v_2 = 6.23\text{ m/s}.$$

**Note 1:** <u>Important point</u>: Notice the angle had nothing to do with this problem. As far as the concept of energy is concerned, it does not matter whether the body is moving downward or upward or sideways. The amount of energy the body has at a given instant is solely related to the body's mass and velocity, NEVER ITS DIRECTION. As such, do not waste time breaking *velocity vectors* into their component parts. All you need is the velocity's *magnitude*.

**Note 2:** You could just as easily have taken ground level to be the *zero potential energy level*. If you had, the initial potential energy would have been $mgy_1$ instead of $mg(y_1 - y_2)$ and the final potential energy would have been $mgy_2$ instead of zero. Both ways work (if you don't believe me, try it); there is no preferred way to attack the problem.

**4.)** <u>Example You've Already Seen, Done the Easy Way:</u> A 2 kilogram block on a horizontal surface is placed without attachment against a spring whose spring constant is $k = 12\ nt/m$. The block is made to compress the spring .5 meters (see Figure 6.21 on the next page). Once done, the block is released and accelerates out away from the spring. If it slides over 2 meters of *frictionless surface* before sliding onto a *frictional* surface, and if it then proceeds to travel an *additional* 13 meters on the *frictional surface* before coming to rest, how large is the *frictional force* that brought it to rest?

**Note:** All the information concerning the *energy state of the system* when the block is <u>at *Point 1*</u> is shown on the sketch. The same is true for *Point 2*. Even the *work done by forces not accommodated by potential energy functions* is written onto the sketch. All the information you need to use the *modified conservation of energy* expression is laid out in its entirety. All that has to be done from there is to put the information into the *c. of e.* equation.

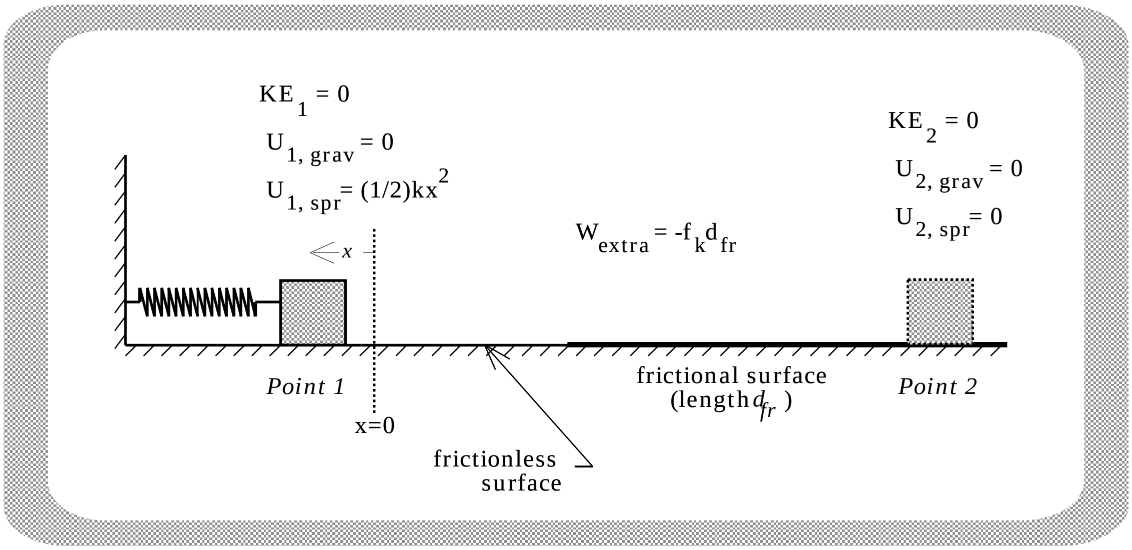


FIGURE 6.21

**a.)** According to the *modified conservation of energy* expression:

$$
\begin{array}{ccccccc}
KE_1 & + & \Sigma\, U_1 & + \Sigma\, W_{ext} = & KE_2 & + & \Sigma\, U_2 \\
(1/2)mv_1^{\ 2} & + & [U_{1,gr} + U_{1,sp}] & + [W_{f_k}] = & (1/2)mv_2^{\ 2} & + & [U_{2,gr} + U_{2,sp}] \\
0 & + & [\ 0 + (1/2)kx^2] & + [-f_k d_{fr}] = & 0 & + & [\ 0 + 0\ ]
\end{array}
$$

$$
\begin{aligned}
\Rightarrow \quad f_k &= [\ k \quad x^2\ ] / [2 \quad d_{fr}\ ] \\
&= [(12\ nt/m)(.5\ m)^2] / [2(13\ m)] \\
&= .115\ \ nts.
\end{aligned}
$$

**b.)** When this example was done in the *work/energy* section, you were told not to get too attached to the *work/energy* approach. Why? Because another approach was coming that was purported to be easier to use.

You have now seen the other technique--the *modified conservation of energy* approach. What makes it so easy? It is primarily end-point dependent. Indeed, you have to manually determine the amount of work done on the body in-between the end-points if you have forces for which you haven't *potential energy functions*, but that is considerably easier than hassling with *work* calculations for *each* force on an individual basis.

Bottom line: In short, the *modified conservation of energy* approach is easier to execute. Get to know it, understand it, practice it, and you'll learn to love it!

**5.)** <u>A More Complex Example</u>: A block of mass $m$ is pressed against an unattached spring whose equilibrium position is $d_1 = 3$ *meters* above ground and whose spring constant is $k = 25.6mg/d_1$ (see Figure 6.22). The block is made to compress the spring a distance $d_1/8$ *meters*. The block is additionally forced against the side-wall by your little sister. The force she applies ($\boldsymbol{F}_{sis}$) has a magnitude of $mg/4$ (no, $mg$ does *not* stand for milligrams; it is the *weight of the block--mass* times *gravity*) at an angle of $60^o$ with the vertical. The wall is *frictional* with a coefficient of friction of $\mu_k = .4$ (see Figure 6.23 below). Once released by *you* (your sister is still pushing), the block falls. How fast will it be traveling when it reaches *Position 2* a distance $d_1/4$ from the ground?

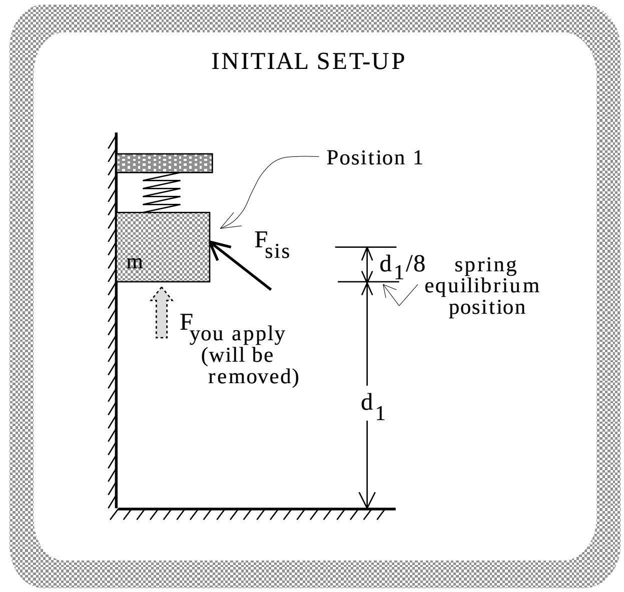


FIGURE 6.22

**a.)** We need an equation that will allow us to determine the velocity of the block after it has moved to $y = d_1/4$. As the *conservation of energy* approach is related to <u>distances traveled</u> (these are wrapped up in the *work* calculations and *potential energy* functions) and velocities (these are wrapped up in the *kinetic energy* calculations), we will try to use that approach here.

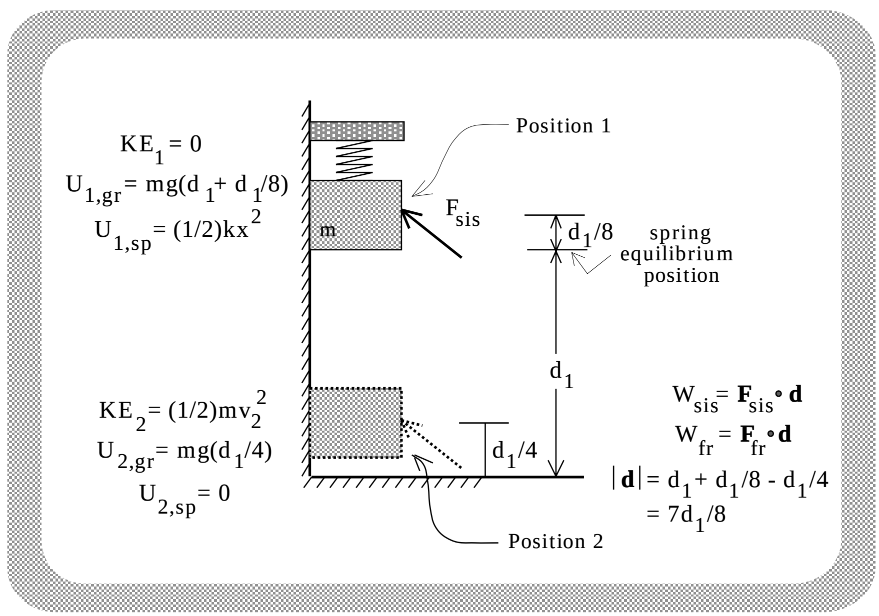


FIGURE 6.23

**Note 1:** As all our *distance measurements* are relative to the ground, we might as well take the *zero potential energy level* for gravity to be at ground-level.

**Note 2:** When the block is released, it is accelerated downward by *gravity* and the *spring* but is also retarded in its acceleration by *friction* and *your little sister*. We know *potential energy* functions for *gravity* and the *spring*, but we have no function for *your sister's force* or *friction*.

**b.)** In its bare bones form, the *modified conservation of energy* equation yields (justification for each part is given below in *Section 5c*):

$$
\begin{array}{lllllll}
KE_1 + & \Sigma U_1 & + & \Sigma W_{ext} & = & KE_2 + & \Sigma U_2 \\
0 + [\ U_{1,gr} & + \ U_{1,sp}\ ] & + [ & W_{sis} \ + \ W_{f_k}\ ] & = & (1/2)mv_2^{\,2} + & [U_{2,gr} + 0] \\
0 + [mg(d_1 + d_1/8) + & (1/2)kx^2\ ] & + [ & \mathbf{F}_{sis}\cdot\mathbf{d}_{sis} \ + \ (-f_k)(d_{fr})\ ] & = & (1/2)mv_2^{\,2} + & [mg(d_1/4) + 0] \\
0 + [\ mg(9d_1/8) & +.5(25.6mg/d_1)(d_1/8)^2] & + [ & (mg/4)(7d_1/8)\cos 120^\circ + (-\mu_k N)(7d_1/8)] & = & (1/2)mv_2^{\,2} + & [.25mgd_1 + 0] \\
0 + [(1.125mgd_1) + & (.2mgd_1)\ ] & + [ & (-.11mgd_1) \ + \ (-.074mgd_1)\ ] & = & (1/2)mv_2^{\,2} + & [.25mgd_1]
\end{array}
$$

$$
\begin{aligned}
\Rightarrow \quad v_2 &= [1.78\ gd_1]^{1/2} \\
&= [1.78\ (9.8\ m/s^2)\ (3\ m)]^{1/2} \\
&= 7.23\ m/s.
\end{aligned}
$$

**c.)** If the pieces used in the above expression are obvious, skip this section and continue onward. If they are not obvious, the following should help:

**i.)** At *Point 1*, as the block is not initially moving:

$$KE_1 = 0.$$

**ii.)** At *Point 1*, the block has *gravitational potential energy*

$$U_{1,gr} = mg(d_1 + d_1/8) = 1.125mgd_1$$

and *spring potential energy*

$$U_{1,sp} = (1/2)kx^2 = (1/2)(25.6mg/d_1)(d_1/8)^2 = .2mgd_1.$$

**iii.)** At *Point 2*, the block has gravitational potential energy

$$U_{2,gr} = mg(d_1/4) = .25mgd_1.$$

**iv.)** At *Point 2*, the block has no *spring potential energy* (as the spring *exerts no force on the block* when the block is at *Point 2*, the spring provides *no potential energy* to the block when at that point):

$$U_{2,sp} = 0.$$

**v.)** At *Point 2*, the block will have *kinetic energy*

$$KE_2 = (1/2)mv_2^{\,2}.$$

**vi.)** In between *Points 1* and *2*, "extraneous" work is done by *little sister* in the amount of:

$$\begin{aligned} W_{sis} &= \mathbf{F}_{sis} \cdot \mathbf{d} \\ &= |\mathbf{F}|\,|\mathbf{d}| \cos\phi \\ &= (F_{sis})\,(d) \cos 120^{o} \\ &= (mg/4)(7d_1/8)(-.5) \\ &= -.11\ mgd_1. \end{aligned}$$

**vii.)** In between *Points 1* and *2*, "extraneous" work is done by friction in the amount of:

$$\begin{aligned} W_{f_k} &= \mathbf{f}_k \cdot \mathbf{d} \\ &= |\mathbf{F}|\,|\mathbf{d}| \cos\phi \\ &= (f_k)\,(d) \cos 180^{o} \\ &= f_k(7d_1/8)(-1) \\ &= -.875 f_k d_1. \end{aligned}$$

**viii.)** To solve this, we need $f_k$. The easiest way to determine $f_k$ is with Newton's Second Law (the *free body diagram* shown in Figure 6.24 is for the body in mid-flight--it looks a bit different from the fbd for the section of flight during which the spring is still engaged, but the horizontal components are identical in both cases). Doing so yields:

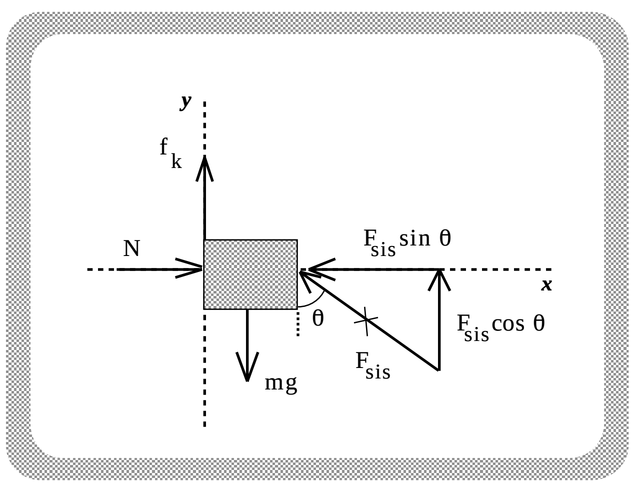


FIGURE 6.24

$\underline{\sum F_x:}$

$$N - F_{sis} \sin\theta = ma_x = 0$$
$$\Rightarrow \quad N = F_{sis} \sin\theta$$
$$= (mg/4) \sin 60$$
$$= .215 \text{ mg}.$$

The frictional force is, therefore:

$$f_k = \mu_k N$$
$$= (.4)(.215 \text{ mg})$$
$$= .085 \text{ mg}.$$

With that, we can determine the work friction does:

$$W_{f_k} = -.875 f_k d_1$$
$$= -.875 (.085 \text{ mg})\, d_1$$
$$= -.074 \text{ mgd}_1.$$

**d.)** As we did in the beginning, putting it all together yields:

$$KE_1 + \sum U_1 + \sum W_{ext} = KE_2 + \sum U_2$$
$$0 + [U_{1,gr} + U_{1,sp}] + [W_{sis} + W_{f_k}] = (1/2)mv_2^2 + [U_{2,gr} + 0]$$
$$0 + [mg(d_1 + d_1/8) + (1/2)kx^2] + [\mathbf{F}_{sis} \cdot \mathbf{d}_{sis} + (-f_k)(d_{fr})] = (1/2)mv_2^2 + [mg(d_1/4) + 0]$$
$$0 + [mg(9d_1/8) + .5(25.6mg/d_1)(d_1/8)^2] + [(mg/4)(7d_1/8)\cos 120^\circ + (-\mu_k N)(7d_1/8)] = (1/2)mv_2^2 + [.25mgd_1 + 0]$$
$$0 + [(1.125mgd_1) + (.2mgd_1)] + [(-.11mgd_1) + (-.074mgd_1)] = (1/2)mv_2^2 + [.25mgd_1]$$

$$\Rightarrow \quad v_2 = [1.78\, gd_1]^{1/2}$$
$$= [1.78\,(9.8 \text{ m/s}^2)(3 \text{ m})]^{1/2}$$
$$= 7.23 \text{ m/s}.$$

## L. One More Twist–Energy Considerations with Multiple-Body Systems:

**1.)** The idea behind the *modified conservation of energy* equation is that it is possible to keep track of not only the amount of energy in a system, but also how the energy is distributed throughout the system.

**2.)** Up until now, all we have examined have been single-body systems. It is possible to extend the *energy considerations* approach to take into account the energy of a whole group of objects.

**3.)** Executing this expanded version of the *modified conservation of energy* approach:

**a.)** Calculate the *total kinetic energy* (i.e., the kinetic energy of each body in the system added together) at time $t_1$.

**b.)** To that, add the *total potential energy* (i.e., all potential energy of all sorts acting on each body in the system, all added together) at time $t_1$.

**c.)** To that, add the *total extra work* done on all the bodies in the system between times $t_1$ and $t_2$.

**d.)** Put the above sum equal to the *total kinetic energy* plus the *total potential energy* in the system at time $t_2$ .

**4.)** The modified *modified conservation of energy equation* thus becomes:

$$\sum KE_{1,tot} + \sum U_{1,\,tot} + \sum W_{extra,tot} = \sum KE_{2,tot} + \sum U_{2,tot}.$$

**5.)** Example: An Atwood Machine is simply a string threaded over a pulley with a mass attached to each end (see Figure 6.25). Assuming the pulley is ideal (i.e., massless and frictionless) and that $m_1 < m_2$, how fast will $m_1$ be moving if the system begins from rest and freefalls a distance $h$ meters?

**a.)** The system in its initial state is shown in Figure 6.25. Notice that each body is assigned a *zero gravitational-potential-energy level* of its own.

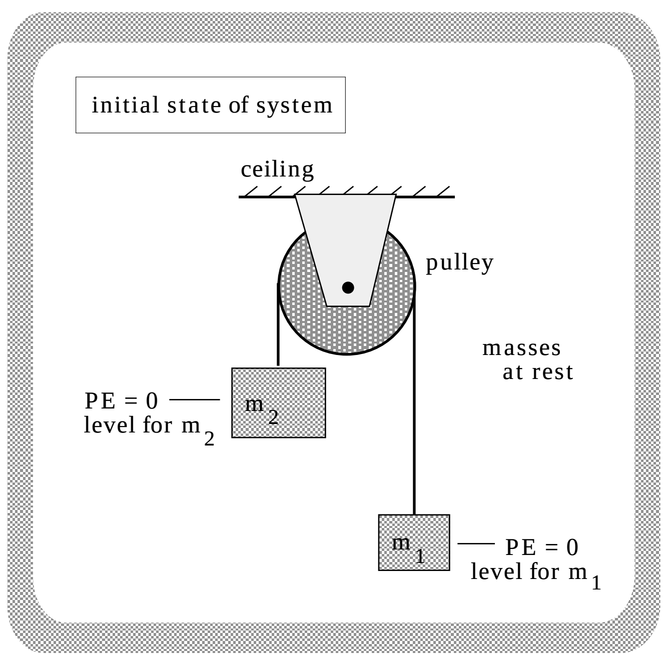


FIGURE 6.25

**Note:** We could have assigned a common level, but it is easier the other way (remember, *where* the zero is for a given body doesn't matter--it is *changes* in *potential energy* that count).

**b.)** Figure 6.26 shows the system after the freefall. Notice that mass $m_2$ has moved *below* its zero-potential-energy-level, making the *potential energy* at that point *negative.*

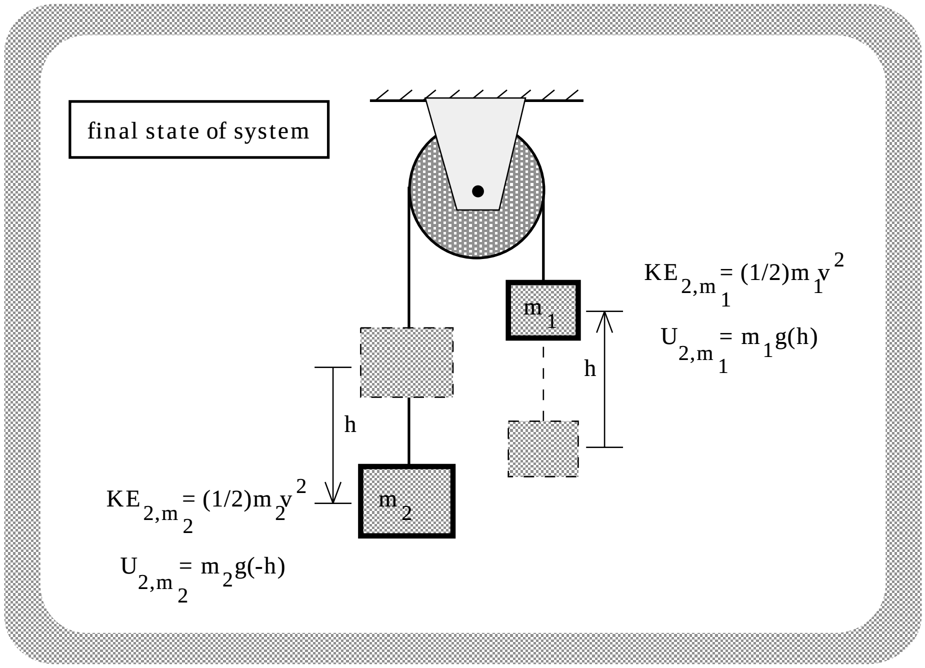


FIGURE 6.26

**Note:** The amount of *work* tension does on $m_1$ is $+T(h)$, whereas the amount of *work* tension does on $m_2$ is $-T(h)$. As such, the two *work quantities* associated with the *tension in the line* add to zero.

**c.)** Putting everything together and executing the *modified conservation of energy* approach, we get:

$$\sum KE_{1,tot} \quad + \quad \sum U_{1,\,tot} \quad + \quad \sum W_{extra,tot} \quad = \quad \sum KE_{2,tot} \quad + \quad \sum U_{2,tot}.$$

$$[KE_{1,m_1} + KE_{1,m_2}] + [U_{1,m_1} + U_{1,m_2}] + [T(h) + T(-h)] = [KE_{2,m_1} + KE_{2,m_2}] + [U_{2,m_1} + U_{2,m_2}]$$

$$[\,0 + 0\,] \quad + \quad [\,0 + 0\,] \quad + \quad [\,0\,] \quad = [.5m_1v^2 + .5m_2v^2] + [m_1gh + m_2g(-h)]$$

$$\Rightarrow \quad v = [[-m_1gh + m_2gh] / [.5(m_1 + m_2)]]^{1/2}.$$

**M.) Power:**

**1.)** There are instances when knowing how much work is done by a force is not enough. As an example, it may seem impressive to know that a particular motor can do 120,000 joules of work, but not if it takes ten years for it to do so. The amount of *work* *<u>per unit time</u>* being done is often more important than *how much* work can be done.

**2.)** The physics-related quantity that measures "work per unit time" is called *power*. It is defined as:

$$P = W/t,$$

where $t$ is the time interval over which the work $W$ is done.

**3.)** The units for power in the MKS system are $kg{\cdot}m^2/s^3$. This is the same as a *joules/second,* which in turn is given the special name *watts*. Although the watt is a unit most people associate with electrical devices (the light bulb you are using to read this passage is probably between 60 watts and 150 watts), the quantity is also used in mechanical systems. Automobiles are rated by their *horsepower*. One horsepower is supposedly the amount of *work* a "standard" horse can do *per unit time*. As *formally* defined, one horsepower equals 746 watts.

**4.)** A special relationship is often derived in physics books that relates the amount of power provided by a force $\boldsymbol{F}$ as it is applied to a body that moves a distance $\boldsymbol{d}$ with constant velocity $\boldsymbol{v}$. Simply presented:

$$\begin{aligned} P_F &= W/t \\ &= (\mathbf{F}\cdot\mathbf{d})/t \\ &= \mathbf{F}\cdot(\mathbf{d}/t) \\ &= \mathbf{F}\cdot\mathbf{v}. \end{aligned}$$

This manipulation has been included for the sake of completeness.

# QUESTIONS

**6.1)** A 3 kilogram mass moving at 2 m/s is pulled 35 meters up a $25^o$ incline by a force $F$ (see Figure I). If the coefficient of friction between the mass and the incline is .3:

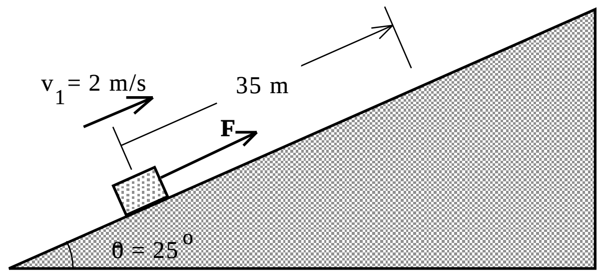


FIGURE I

**a.)** How much work does *gravity* do as the mass moves up the incline to the 35 meter mark?

**b.)** How much work does *friction* do as the mass moves up the incline to the 35 meter mark?

**c.)** How much work does the *normal force* do as the mass moves up the incline to the 35 meter mark?

**d.)** How much *kinetic energy* does the mass initially have?

**e.)** WORK/ENERGY PROBLEM: Assuming the mass's *velocity* at the 35 meter mark is 7 m/s, use the *work/energy theorem* to determine the force $F$. *Do this as you would on a test.* That is, forget for the moment that you have done any work above and lay this problem out completely in algebraic form before putting in numbers.

**6.2)** A force $\boldsymbol{F}$ is applied to a mass $m = .5\ kg$ as it proceeds up a frictional, hemispherical dome ($\mu_k = .2$) of radius $R = .3$ *meters*. The force is ALWAYS at an angle of $12^o$, relative to the mass's motion (see Figure II). $\boldsymbol{F}$ also varies so as to keep the mass's very slow velocity constant throughout the motion. Assuming the velocity is additionally very, very small (i.e., so that the centripetal acceleration $v^2/R$ is minuscule), how much work does $\boldsymbol{F}$ do on the mass as it moves from an angle of $20^o$ to an angle of $60^o$ up the dome? Put the numbers in last.

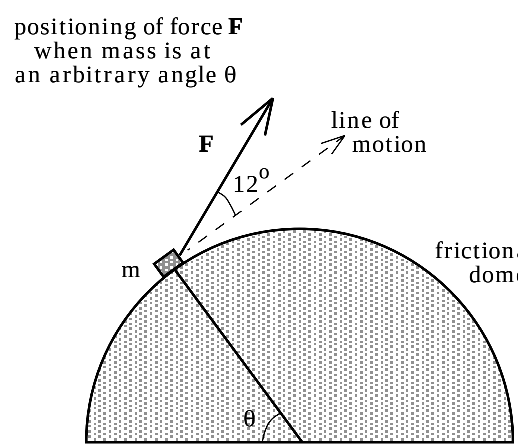


FIGURE II

**6.3)** How much energy is stored in a spring compressed 20 centimeters (.2 meters) if the spring's *spring constant* is $k=120\ nts/m$?

**6.4)** Derive an expression for the force function associated with the *potential energy* function shown below.

$$U(x,y) = -(k_1/x)e^{-ky}.$$

You may assume the *k* terms have magnitudes of *one* and have the appropriate units.

**6.5)** Determine the *potential energy function* for the force function:

$$\mathbf{F} = [(k_1 \ln x) - 3]\mathbf{i} - [k_2 y^2]\mathbf{j}.$$

Assume the magnitudes $k_1$ and $k_2$ are both *one* and that each has the appropriate units. Also, assume that $x > 1$.

**6.6)** Tarzan ($m_T = 80\ kg$) stands on a 12 meter high knoll (see Figure III). He grabs a taut, 15 meter long vine attached to a branch located 17 meters above the ground and swings down from rest to Jane perched on a 5 meter high mole hill (they breed particularly big moles in Africa).

**a.)** What is Tarzan's velocity when he reaches Jane?

**b.)** What is the tension in the vine when Tarzan is at the bottom of the arc? (Note: Tarzan is moving through a CIRCULAR path).

**c.)** What is the tension in the vine just before Tarzan lets go upon reaching Jane?

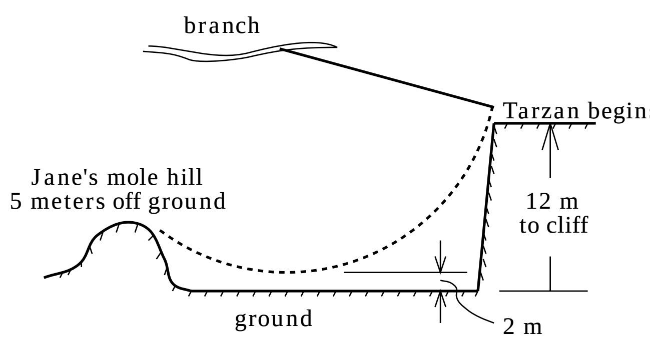


FIGURE III

**6.7)** A 12 kilogram crate starts from rest at the top of a curved incline whose radius is 2 meters (see Figure IV). It slides down the incline, then proceeds 18 additional meters before coming to rest. What is the *frictional force* between the crate and the supporting floor (both curved and horizontal)?

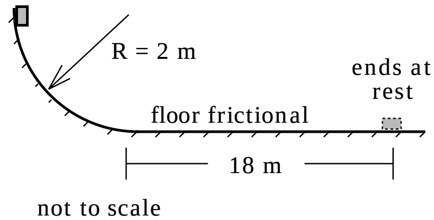


FIGURE IV

Assume this *frictional force* is constant throughout the entire motion.

**6.8)** Pygmies use blow-guns and 15 gram (.015 kg) darts dipped in the poison curare to immobilize and kill monkeys that live in the tree-top canopy of their forest home. Assume a pygmy at ground level blows a dart at $85^o$ (relative to the horizontal) at a monkey that is 35 vertical meters up (over 100 feet). Assuming a dart must be moving at 4 m/s to effectively pierce monkey skin, what is the *minimum velocity* the dart must be moving as it leaves the blow-gun if it is to pierce the monkey?

**6.9)** A freewheeling 1800 kilogram roller coaster cart is found to be moving 38 m/s at *Point A* (see Figure V). The actual distance between:

*Point A* and *Point B* is 70 meters;
*Point B* and *Point C* is 60 meters;
*Point C* and *Point D* is 40 meters.

If the average frictional force acting throughout the motion is 27 newtons, the radius of the loop is 20 meters, the first hill's height 25 meters, the first dip 15 meters, and the incline just after the loop coming directly off the loop's bottom at an angle of $30^o$:

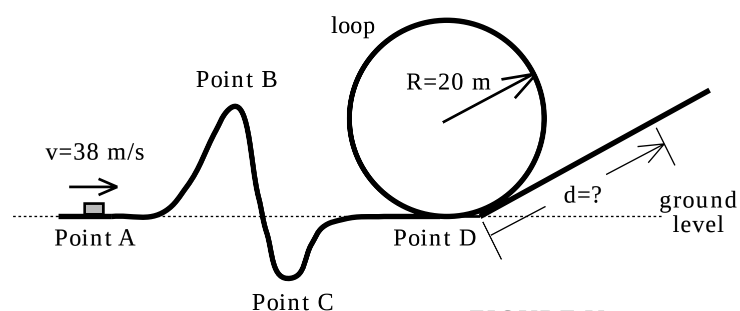


FIGURE V

**a.)** How fast is the cart moving at *Point C*?

**b.)** How far up the incline *d* will the cart travel before coming to rest?

**c.)** What must the cart's *minimum velocity* be at *Point A* if it is to just make it through the top of the loop without falling out of its CIRCULAR MOTION (hint, hint).

Note: The phrase "just making it through the top" means that for all intents and purposes, the *normal force* applied to the cart by the track goes to zero leaving gravity the only force available to affect the cart's motion at the top.

**6.10)** A spring-loaded bumper is placed on a $55^o$ frictional incline plane. A 60 kilogram crate breaks loose a distance 3 meters up the incline above the bumper and accelerates down the incline (see Figure VI). If the average *frictional force* applied to the crate by the incline is 100 newtons and the spring constant is 20,000 newtons/meter:

**a.)** How much will the bumper spring compress in bringing the crate to rest? (Assume there is friction even after the crate comes in contact with the bumper).

**b.)** The crate compresses the bumper's spring which then pushes the crate back up the incline (the crate effectively bounces off the bumper). If a total of three-quarters of the crate's *kinetic energy* is lost during the collision, how far back up the incline will the crate go before coming to rest?

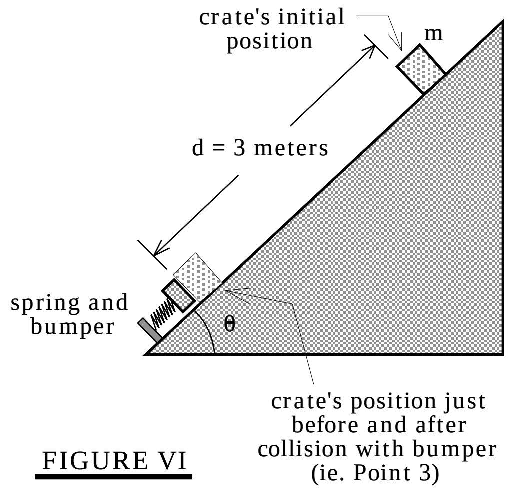


FIGURE VI

**6.11)** Because *gravitational attraction* between you and the earth becomes less and less as you get higher and higher above the earth's surface, the *gravitational potential energy function* for a body of mass $m_1$ that is a substantial distance *d units* away from the earth's surface is not *mgh*; it is:

$$U = -\,Gm_1 m_e/(r_e + d),$$

where *G* is called *the universal gravitational constant* ($6.67\text{x}10^{-11}$ $m^3/kg\cdot s^2$), $m_e$ is the mass of the earth ($5.98\text{x}10^{24}$ kg), and $r_e$ is the radius of the earth ($6.37\text{x}10^6$ m).

A satellite is observed moving at 1500 m/s when 120,000 meters above the earth's surface. It moves in an elliptical path which means its height and velocity are not constants. After a time, the satellite is observed at 90,000 meters. Ignoring frictional effects, how fast is the satellite traveling at this second point?

**6.12)** A string of length *L* is pinned to the ceiling at one end and has a mass *m* attached to its other end. If the mass is held in the horizontal and released from rest, it freefalls down through an arc of radius *L* until the string collides with a peg located a distance *L/3* from the bottom of the arc (see Figure VIIa). From there it proceeds along an arc of lesser radius (i.e., a radius of *L/3*).

Assuming one-tenth of the energy in the system is lost during this collision, what will the tension $T$ in the string be as the body moves through the *top* of its final arc (see Figure VIIb)?

INITIAL POSITION

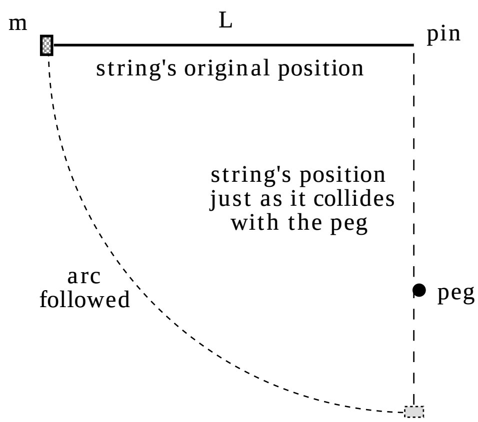


FIGURE VIIa

AT TOP OF SWING

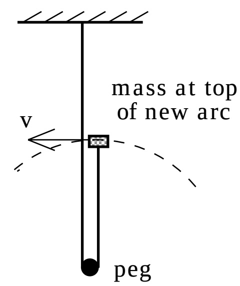


FIGURE VIIb